Annex A – GPS FUNDAMENTALS

A.1 GENERAL

The NAVSTAR[1] Global Positioning System (GPS) is a space-based radio navigation system, which was designed by the United States Department of Defense (DoD) for the US armed forces. The US Air Force manages the program at the GPS Joint Program Office (JPO). The Defence Mapping Agency (DMA), Department of Transport (DOT), and other US military services as well as the North Atlantic Treaty Organisation (NATO) have representatives at the JPO.

The GPS provides suitably equipped users with highly accurate Position, Velocity and Time (PVT) data. This service is provided globally, continuously, and under all weather conditions to users at or near the surface of the earth. GPS receivers operate passively, thereby allowing an unlimited number of simultaneous users. GPS has features that can deny accurate service to unauthorised users, prevent spoofing and reduce receiver susceptibility to jamming.

In this Annex, only a brief introduction to GPS is given, with emphasis on characteristics relevant to this dissertation. Further information about the GPS technical features and specific applications can be found in the copious technical literature available on the subject, such as references [1 – 6].

A.2 GPS SEGMENTS

The GPS comprises 3 major segments, Space, Control and User Segments. The Space Segment consists of a constellation of GPS satellites in semi-synchronous orbits around the earth. Each satellite broadcasts radio-frequency ranging codes and a navigation data message. The Control Segment consists of a Master Control Station (MCS) and a number of monitor stations located around the world.

The MCS is responsible for tracking, monitoring and managing the satellite constellation and updating the navigation data messages. The User Segment consists of an unlimited number of users (civilian and military) equipped with a variety of GPS receivers specifically designed to process the satellite signals. The Space, Control and User Segments are briefly described in the following paragraphs.

A.2.1 Space Segment

The basic GPS Space Segment consists of 24 operational satellites. The satellites are placed in 6 orbital planes with 4 operational satellites in each plane. The satellite orbital planes have an inclination, relative to the equator, of 55° and the orbit height is of about 20,200 km. The satellites complete an orbit in approximately 12 hours. The relative phasing of satellites from one orbital plane to the next is 40°.

The satellites are positioned such that a minimum of 4 satellites are observable by a user anywhere on the earth and each satellite will be observable for approximately 5 hours at a time. Each satellite is equipped with four atomic clocks in order to measure time and to produce the fundamental frequency f = 10.23 MHz. Multiplying this fundamental frequency by 154 gives 1575.42 MHz, which is the frequency of the L1 carrier; and multiplying by 120 gives 1227.60 MHz, which is the frequency of the L2 carrier.

The GPS satellites continuously broadcast on two L-band frequencies (L1 and L2). Superimposed on these carriers are two coded signals unique to each satellite: a precision code (P-code) Pseudo Random Noise (PRN) signal with a 10.23 MHz chip rate (i.e., f) and a coarse/acquisition code (C/A code) PRN signal with 1.023 MHz chip rate (i.e., $f/10$). The L1 frequency contains both the P-code and C/A code while the

[1] NAVigation Signal Time And Range.

L2 frequency contains the P code only. Furthermore, the 'navigation message' is modulated on both carriers at a chipping rate of 50 Hz. In general, the signal structure can be described by the following equation:

$$S_{GPS}(t) = \underbrace{A_C C(t) D(t) \sin(2\pi f_{L1} t) + A_P P(t) D(t) \cos(2\pi f_{L1} t)}_{L1 \quad Signal} + \underbrace{B_P P(t) D(t) \cos(2\pi f_{L2} t)}_{L2 \quad Signal} \tag{A.1}$$

where A_C, A_P, and B_P denote the amplitudes of the C/A-code on L1, of the P-code on L2 and of the P-code on L2 respectively. C(t) is the C/A code, P(t) is the P-code, and D(t) is the navigation message.

A.2.2 Control Segment

The GPS Control Segment consists of one Master Control Station (MCS) at Falcon AFB in Colorado Springs (USA), plus five monitor stations at the MCS, Hawaii, Kwajalein, Diego Garcia and Ascension. All monitor stations except Hawaii and Falcon are also equipped with ground antennae for communications with the GPS satellites. The monitor stations passively track all GPS satellites in view, collecting ranging data from each satellite. This information is passed on to the MCS where, after extensive computations, the satellite ephemeris and clock parameters are estimated and predicted. The ephemeris and clock data are up-loaded via any of the ground antennae to the satellite, on a S-band link, for retransmission in the navigation message.

The satellite clock drift is corrected so that all transmitted data are synchronised with GPS time (Master Time). The fundamental equation is the following:

$$t_{GPS} = t_s - \Delta t_s \tag{A.2}$$

where:

t_{GPS}	=	GPS time;
t_s	=	satellite time; and
Δt_s	=	difference between satellite and GPS time.

The corrections are applied to the last term of equation (A.2) using 3 polynomial coefficients and 1 relativistic correction term.

The correction equation is therefore:

$$\Delta t_s = a_0 + a_1\left(t_{GPS} - t_{0c}\right) + a_2\left(t_{GPS} - t_{0c}\right)^2 + \Delta t_r \tag{A.3}$$

where:

a_0, a_1, a_2	=	polynomial coefficients for phase, frequency and age offset;
Δt_r	=	relativistic correction term; and
t_{0c}	=	time of transmission of the corrections.

The correction terms are all estimated in the MCS, up-loaded to each satellite and then transmitted in the navigation message.

The ephemeris corrections are obtained through the estimation of the Cartesian co-ordinates of the satellites along the orbits by integrating there motion equations.

A.2.3 User Segment

The User Segment consists of a variety of military and civilian GPS receivers specifically designed to receive, decode and process the satellite signals. They include stand-alone receiver sets, as well as equipment that is integrated with or embedded into other systems. They serve a variety of user applications including navigation, positioning, time transfer, surveying and attitude reference. Consequently, GPS receivers for different applications can vary significantly in design and function.

In general, GPS receivers can be divided into two major groups, those that can track four or more satellites simultaneously (multi-channel receivers), and those that can scan or sequence between all visible satellites. Sequencing receivers can be divided into the following additional categories:

- Single-channel;
- Fast-multiplexing single-channel; and
- Two-channel.

Multi-channel receivers, available in a variety of forms, allow simultaneous reception of four, six, eight, ten or twelve channels. These devices are essential in dynamic applications that require accurate, instantaneous position and velocity information. Multi-channel devices have one strong advantage in being able to track as many satellites as channels available, enabling it to choose the most appropriate satellites to obtain the lowest possible GDOP figure. More detailed information about GPS receiver's technology can be found in the references [1 – 6].

A.3 GPS POSITIONING SERVICES

The GPS services include the Standard Positioning Service (SPS) and the Precise Positioning Service (PPS). For military user community and certain selected civil users, the requirements for the PPS are: 16 metres spherical error probable (SEP) for position accuracy, less than 10 cm/sec for velocity accuracy, and less than 100 nsec difference related to the time transfer between GPS Time and Universal Time Co-ordinated (UTC). Demonstrations and actual tests have shown that the above system level requirements are easily met.

The GPS SPS requirements for the civil and military user community are: 100 m 2D-RMS (approximately 2 sigma value or 95 %) for horizontal position accuracy, 175 m (2 sigma value) for vertical accuracy, less than 0.3 m/sec velocity accuracy, and less than 220 nanoseconds difference related to the time transfer between GPS and UTC.

The SPS is the result of Selective Availability (SA) of the PPS (i.e., to deny full GPS accuracy to unauthorised users). SA was first implemented on the 25th of March 1990. This is obtained by implementing two different features: *Dither* and *Epsilon*. Dither is the result of introducing drifts in the satellite clocks and Epsilon is the degradation of the accuracy of the transmitted ephemeris.

The PPS is based on the P code, which was modified to a classified Y code beginning on the 31 January 1994. Thus, PPS is available only to authorised users with decryption capability. The modification of P code to Y code is called Anti-Spoofing (AS) and was introduced by the US Department of Defense to prevent hostile imitation of the PPS.

For many applications the SA degradation of accuracy can be overcome or reduced by adopting differential techniques and by using carrier phase observations in addition to code observations [7].

A.4 GPS OBSERVABLES

There are basically three types of GPS observables: pseudorange, carrier phase, and Doppler observable. Pseudoranges are commonly used in navigation and can provide an accuracy ranging from 100 m (stand alone C/A code positioning), to 2 m in Differential GPS (DGPS) positioning. The carrier phases are traditionally used in high precision surveying and can achieve sub-centimetre accuracy. However, it is most common to use combination of pseudoranges and carrier phases. Moreover, it has become common practice to take advantage of various combinations of the original phase observation, such as double differences and triple differences.

The various DGPS techniques, including carrier phase applications, are detailed in the Chapter 1 of this AGARDograph. An introduction to the various GPS observables is given in the following paragraphs.

A.4.1 Pseudorange Observable

The concept of pseudoranging is based on measuring difference between the time of transmission of the code from the satellite and the epoch of reception of the same signal at the receiver antenna. This is achieved by correlating identical Pseudorandom Noise (PRN) codes generated by the satellite's clock, with those generated internally by the receiver's own clock. If this time difference is multiplied by the speed of propagation of the radio wave, a range value is obtained which is the distance between the satellite and the receiver's antenna referring to the epoch of observation. Both receiver and satellite clock errors affect the pseudoranges. Therefore, they differ from the actual geometric distance corresponding to the epochs of emission and reception. The general pseudorange equation is:

$$P_k^p\left(t_k\right) = \left(t_A - t^i\right) \cdot c \tag{A.4}$$

where P_k^p represents the actual measurement, t_k denotes the nominal time of the receiver clock ***k*** at reception, t^p denotes the nominal time of the satellite clock ***p*** at emission and ***c*** denotes the speed of light.

Equation (A.4) would correspond to the actual distance between the satellite and receiver's antenna, if there were no clock biases, the signal travelled through vacuum and there was no multipath effect. The clock drifts can be represented by the following expressions:

$$t_{r,k} = t_k + dt_k \tag{A.5}$$

$$t_k^p = t^p + dt^p \tag{A.6}$$

where the symbol ***r*** denotes the true time and the terms dt_k and dt^p are the receiver and satellite clock errors respectively.

Taking these errors and biases into account, the complete expression for the pseudorange becomes:

$$\begin{aligned} P_k^p\left(t_k\right) &= \left(t_{r,k} - t_r^p\right)c - \left(dt_k - dt^p\right)c + I_{k,p}^p\left(t_k\right) + T_k^p\left(t_k\right) \\ &\quad + d_{k,P}\left(t_k\right) + d_{k,P}^p\left(t_k\right) + d_P^p\left(t_k\right) + \varepsilon_p \\ &= \rho_k^p\left(t_{r,k}\right) - \left(dt_k - dt^p\right)c + I_{k,p}^p\left(t_k\right) + T_k^p\left(t_k\right) \\ &\quad + d_{k,P}\left(t_k\right) + d_{k,P}^p\left(t_k\right) + d_P^p\left(t_k\right) + \varepsilon_P \end{aligned} \tag{A.7}$$

where $I_{k,P}^{p}(t_k)$ and $T_k^p(t_k)$ are the ionospheric and tropospheric delays, depending on varying conditions along the path of the signal. The symbols $d_{k,P}(t_k)$ and $d_P^p(t_k)$ denote the receiver and satellite hardware code delays respectively. The symbol $d_{k,P}^{p}(t_k)$ denotes the multipath of the codes, which depends on the geometry of the antenna and satellite with respect to surrounding reflective surfaces. The term ε_p denotes the random measurement noise [5]. The term $\rho_k^p(t_{r,k})$ is the actual geometric distance between the receiver's antenna and the satellite at a specific epoch and therefore:

$$\rho_k^p(t_k) = \sqrt{(u^p - u_k)^2 + (v^p - v_k)^2 + (w^p - w_k)^2} \tag{A.8}$$

The terms (u_k, v_k, w_k) are the approximate Cartesian co-ordinates of the receiver and (u^p, v^p, w^p) denote the position of the satellite at the epoch of transmission and both triplets are expressed in WGS84.

The co-ordinates of the user receiver (and GPS time) can be derived from the simultaneous observation of four or more satellites. Pseudorange measurements can be recorded over many epochs and can be used to perform real-time navigation. The navigation solution is described in the following paragraph.

A.4.1.1 Navigation Solution

For many applications, where high accuracy is not needed (e.g., long and medium range aircraft navigation), the navigation solution based on pseudoranges is appropriate. Most of the errors listed above are not taken into account in this solution. The only unknowns considered in equation (A.7) are the satellite ephemeris, the receiver clock error and the receiver location. The satellite clock error term is assumed negligible, because corrections for it are transmitted to the user as part of the navigation message. The ionospheric and tropospheric delay can be computed (approximately) from ionospheric and tropospheric models. Hardware delays and multipath can be neglected in the navigation solution. Finally, by using multiple-channel receivers specifically designed for navigation, the receiver clock drift and the random measurement noise can be eliminated (as well as most of remaining biases in the system). Thus, the four unknowns in equation (A.7) can be computed with the use of four pseudoranges measured simultaneously to four GPS satellites (Figure A-1).

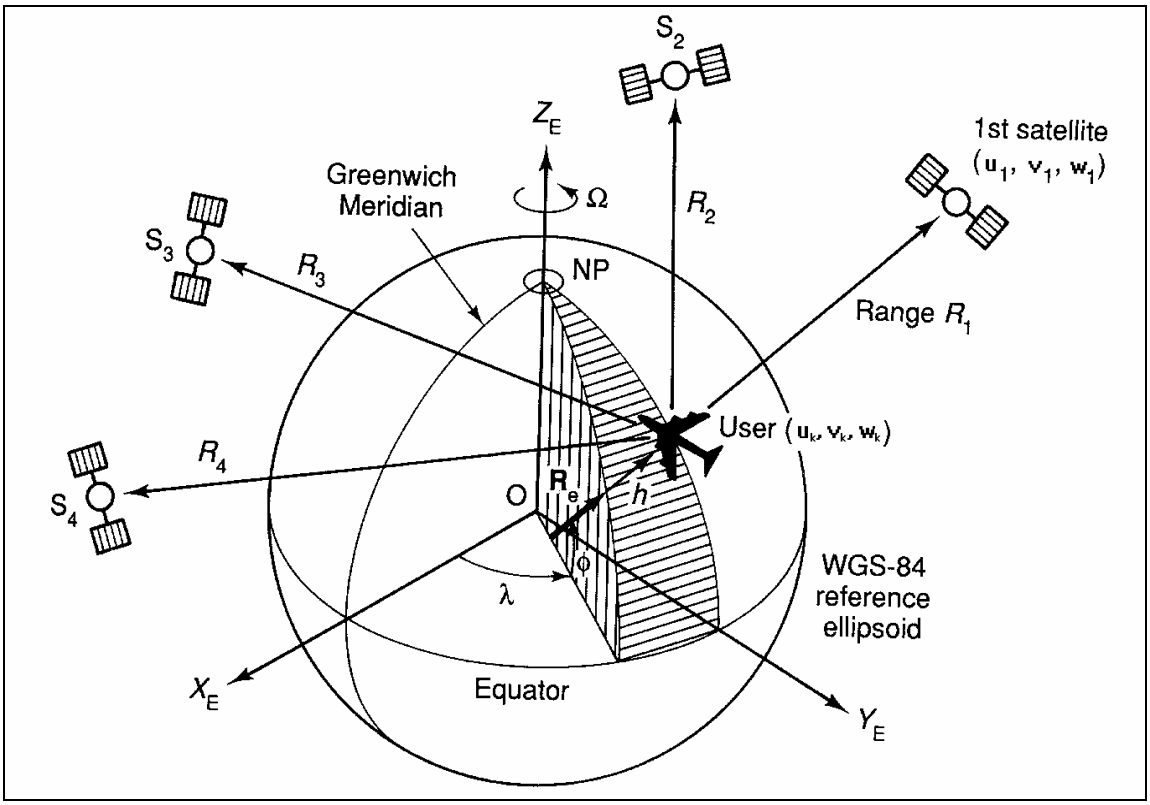

Figure A-1: Navigation Solution in the ECEF Coordinate System (at time zero, the X_e axis passes through the North Pole, and the Y_e. axis completes the right-handed orthogonal system – because of the earth's rotation, the user position is constantly changing in longitude with time).

The following system of equations is therefore formed:

$$\begin{cases} P_k^1(t) = \sqrt{\left(u^1 - u_k\right)^2 + \left(v^1 - v_k\right)^2 + \left(w^1 - w_k\right)^2} - cdt_k \\ P_k^2(t) = \sqrt{\left(u^2 - u_k\right)^2 + \left(v^2 - v_k\right)^2 + \left(w^2 - w_k\right)^2} - cdt_k \\ P_k^3(t) = \sqrt{\left(u^3 - u_k\right)^2 + \left(v^3 - v_k\right)^2 + \left(w^3 - w_k\right)^2} - cdt_k \\ P_k^4(t) = \sqrt{\left(u^4 - u_k\right)^2 + \left(v^4 - v_k\right)^2 + \left(w^4 - w_k\right)^2} - cdt_k \end{cases} \tag{A.9}$$

The receiver clock error dt_k is considered constant for measurements to any satellite.

If more than four satellites are visible, a least-square solution can be performed. The GPS receiver calculates its position in an Earth-Centred Earth-Fixed (ECEF) Cartesian co-ordinate system (WGS-84). These co-ordinates may be expressed to some other system such as latitude, longitude, and altitude if desired. The accuracy provided from the navigation solution is in the order of 5 to 20 m if the P-code is used and approximately 100 m if the C/A code is used.

A.4.1.2 DOP Factors

Ranging error alone does not determine position fix accuracy. The accuracy of the navigation solution is also affected by the relative geometry of the satellites and the user. This is described by the Dilution Of Precision (DOP) factors.

Solution of the system (A.9) requires the measurement of the pseudoranges to four different satellites. The GPS receiver's computer may be programmed to solve directly the navigation equations in the form given above. However, the computation time required in order to solve them, even if only a few seconds, may be too long for many applications [9]. As an alternate way, these equations may be approximated by a set of four linear equations that the GPS receiver can solve using a much faster and simpler algorithm. The system of equations (A.9) can be rewritten in the form:

$$P_k^i = \sqrt{\left(u^i - u_k\right)^2 + \left(v^i - v_k\right)^2 + \left(w^i - w_k\right)^2} + T \quad \text{(i = 1, 2, 3, 4)} \tag{A.10}$$

where u^i, v^i and w^i represent the co-ordinates of the ith satellite, $T = dt_k$ and the units have been chosen so that the speed of light is unity. Linearization of equation (A.10) can proceed as described in Thomis [8] and Siouris [9]. The resulting set of linearized equations relate the pseudorange measurements to the desired user navigation information as well as the user's clock bias:

$$\left(\frac{u_n - u^i}{P_{ni} - T_n}\right)\Delta u_k + \left(\frac{v_n - v^i}{P_{nii} - T_n}\right)\Delta v_k + \left(\frac{w_n - w^i}{P_{ni} - T_n}\right)\Delta w_k + \Delta T = \Delta P_i \quad \text{(i = 1, 2, 3, 4)} \tag{A.11}$$

where:

u_n, v_n, w_n, T_n	=	nominal (a priori best-estimate) values of u_k, v_k, w_k and T;
$\Delta u_k, \Delta v_k, \Delta w_k, \Delta T$	=	corrections to the nominal values;
P_{ni}	=	nominal pseudorange measurement to the ith satellite; and
ΔP_i	=	difference between actual and nominal range measurements.

The quantities on the right-hand side are simply the differences between the actual measured pseudoranges and the predicted measurements, which are supplied by the user's computer, based on knowledge of the satellite position and current estimate of the user's position and clock bias. Therefore, the quantities to be computed ($\Delta u_k, \Delta v_k, \Delta w_k, \Delta T$) are the corrections that the user will make to the current estimate of position and clock time bias. The coefficients of these quantities on the left-hand side represent the direction cosines of the line-of-sight (LOS) vector from the user to the satellite as projected along the Cartesian co-ordinate system.

The four linearized equations represented by equation (A.11) can be expressed in matrix notation as:

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & 1 \\ \beta_{21} & \beta_{22} & \beta_{23} & 1 \\ \beta_{31} & \beta_{32} & \beta_{33} & 1 \\ \beta_{41} & \beta_{42} & \beta_{43} & 1 \end{bmatrix} \times \begin{bmatrix} \Delta u_k \\ \Delta v_k \\ \Delta w_k \\ \Delta T \end{bmatrix} = \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{bmatrix} \tag{A.12}$$

where β_{ij} is the direction cosine of the angle between the LOS to the ith satellite and the jth co-ordinate. This can be written more compactly as:

$$B\bar{x} = r \tag{A.13}$$

where:

B = 4×4 solution matrix (i.e., matrix of coefficients of the linear equation);

$\bar{x}$ = user position and time correction vector ($\bar{x} \equiv [\Delta u_k \quad \Delta v_k \quad \Delta w_k \quad \Delta T]^T$); and

r = pseudorange measurement difference vector ($r \equiv [\Delta P_1 \quad \Delta P_2 \quad \Delta P_3 \quad \Delta P_4]^T$).

It is possible to estimate the covariance of the user position and time correction vector using the equation [10]:

$$\mathrm{cov}(\bar{x}) = B^{-1}\,\mathrm{cov}(r)B^{-T} \tag{A.14}$$

To a good approximation (i.e., assuming statistically independent random variables), Eq. (A.14) can be written as:

$$\mathrm{cov}(\bar{x}) = \left(B^T B\right)^{-1} \tag{A.15}$$

Assuming sufficient signal strength, this covariance matrix depends only on the direction and is no way dependent on the distances between the user and each satellite. The explicit form of the covariance matrix is:

$$\mathrm{cov}(\bar{x}) = C = \begin{bmatrix} \sigma_u^2 & \sigma_{uv} & \sigma_{uw} & \sigma_{ut} \\ \sigma_{vu} & \sigma_v^2 & \sigma_{vw} & \sigma_{vt} \\ \sigma_{wu} & \sigma_{wv} & \sigma_w^2 & \sigma_{wt} \\ \sigma_{tu} & \sigma_{tv} & \sigma_{tw} & \sigma_t^2 \end{bmatrix} \tag{A.16}$$

The diagonal elements of this matrix are actually the variances of user position and time. The various DOP values are obtained as functions of the diagonal elements of the covariance matrix. Converting the Cartesian co-ordinates in matrix (2.16) to more convenient local geodetic co-ordinates, we have:

$$C_{LG} = \begin{bmatrix} \sigma_N^2 & \sigma_{NE} & \sigma_{NH} \\ \sigma_{EN} & \sigma_E^2 & \sigma_{EH} \\ \sigma_{HN} & \sigma_{HE} & \sigma_H^2 \end{bmatrix} \tag{A.17}$$

Table A-1 shows the relationship between the DOP factors and the diagonal elements of the matrixes (A.16) and (A.17).

Table A-1: DOP Expressions

Vertical Vector (1d)	VDOP = σ_H
Horizontal Vector (2d)	HDOP = $(\sigma_N^2 + \sigma_E^2)^{1/2}$
Position Vector (3d)	PDOP = $(\sigma_N^2 + \sigma_E^2 + \sigma_H^2)^{1/2}$
Time, clock offset vector (1d)	TDOP = σ_t
Geometric, Position and Time Vector (4d)	GDOP = $(\sigma_N^2 + \sigma_E^2 + \sigma_H^2 + \sigma_t^2)^{1/2}$

It is also evident that:

$$PDOP = \sqrt{HDOP^2 + VDOP^2} \tag{A.18}$$

$$GDOP = \sqrt{PDOP^2 + TDOP^2} \tag{A.19}$$

The PDOP is very frequently used in navigation. This is because it directly relates error in GPS position to error in pseudo-range to the satellite. There is a proportionality between the PDOP (and GDOP) factor and the reciprocal value of the volume V of a particular tetrahedron formed by the satellites and the user position (Figure A-2).

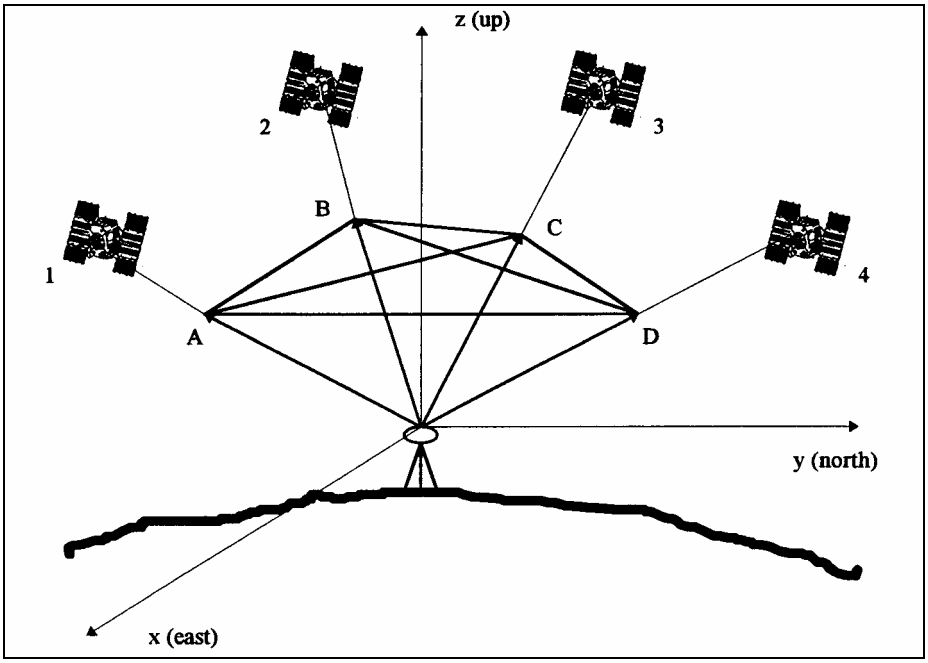

Figure A-2: PDOP Tetrahedron. Four unit-vectors point toward the satellites and the ends of these vectors are connected with 6 line segments.

It is in fact demonstrated [11] that the PDOP is the RSS (i.e., square root of the sum of the squares) of the areas of the 4 faces of the tetrahedron, divided by its volume (i.e., the RSS of the reciprocals of the 4 altitudes of the tetrahedron). Therefore, we can write:

$$PDOP = \sqrt{\frac{1}{h_A^{\,2}} + \frac{1}{h_B^{\,2}} + \frac{1}{h_C^{\,2}} + \frac{1}{h_D^{\,2}}} \tag{A.20}$$

At a particular time and location, the satellites 1, 2, 3, and 4 shown in Figure A-2, have determined elevations (°) and azimuths (°) with respect to the receiver. From these satellite positions the components *(x, y, z)* of the unit vectors to the satellites can be determined. Using the Pythagorean theorem, the distances between the ends of the unit vectors can be determined. Knowing these distances, a tetrahedron can be constructed (Figure A-3) and the four altitudes of the tetrahedron can be measured. Finally, using equation (A.20), the PDOP can be determined. There is no approximation associated with the geometric expression. Any residual error in PDOP determination is only due to construction of the tetrahedron and measurement of the altitudes. This method can be the basis for a computer algorithm designed to solve the 3-dimensional problem of PDOP determination.

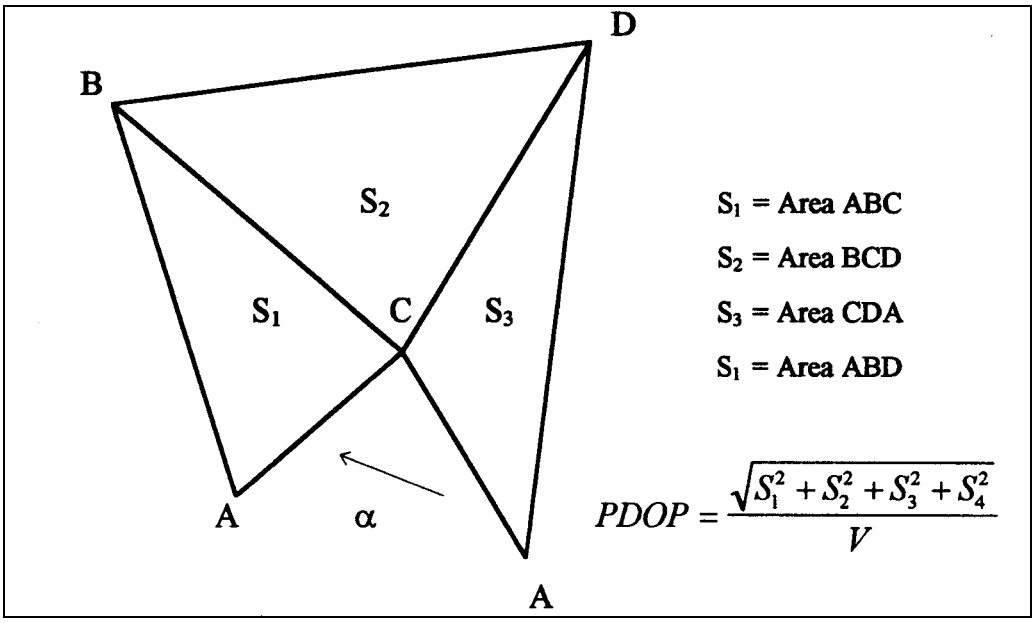

Figure A-3: "Cut and Fold" Tetrahedron for PDOP Determination.

If the angle $A\hat{C}A$ in Figure A-3 is small, one can recognise that the PDOP will be large (poor) even without measuring the altitudes, since this leads to a tetrahedron having a small volume.

From the geometric definition of PDOP we deduce that it is optimal from the user's point of view (i.e., low PDOP) to select the four satellites giving a large volume of the tetrahedron (and a small RSS of the areas of the 4 faces of the tetrahedron). This can be obtained primarily by selecting a combination of satellites far from each other and uniformly distributed around the receiver. The condition for the maximum volume is with a satellite at the zenith and the other three separated by 120° (azimuth) and low over the horizon. This condition, however, would degrade the signal quality, due to the longer propagation paths (i.e., a compromise between signal quality and accuracy of the solution is therefore necessary).

A further consequence of the above construction is that if the ends of the unit vectors are coplanar (a not unusual circumstance) the PDOP becomes infinite, and a position is not obtainable. This is the reason for which the final GPS constellation has been designed so that there are almost always 5 satellites in view anywhere on earth, giving an alternate choice of the 4 satellites to be utilised.

With Block II satellites the DOP values in an open environment are [12]:

50% of time:	HDOP ≤ 1.4;	VDOP ≤ 2.0.	⇨	PDOP ≤ 2.5.
90% of time:	HDOP ≤ 1.7;	VDOP ≤ 2.8.	⇨	PDOP ≤ 3.3.

If m satellites are in view, the number of possible combinations is:

$$N = \frac{m!}{4!(m-4)!} \tag{A.21}$$

In most GPS receivers the number of combinations N also corresponds to the number of PDOP/GDOP computations necessary for selection of the best satellite geometry. Some systems can automatically reject, prior performing positioning calculations, subsets of satellites with associated DOP factors below pre-set thresholds.

A.4.2 Carrier Phase

The carrier phase observable is the difference between the received satellite carrier phase and the phase of the carrier generated by the receiver oscillator. The same error sources that affect pseudoranges, are responsible for the errors which determine the positional accuracy achieved with carrier phases [5]. Clearly, the mathematical formulation of any specific error component is different from the pseudorange case (i.e., phase measurement errors instead of range measurement errors). Since the antenna cannot sense the number of whole carrier waves between the satellite and the receiver (Integer Ambiguity), an extra parameter is inserted in the carrier phase equation:

$$\begin{aligned}\Phi_k^p = \Phi_k(t) - \Phi^p(t) + N_k^p(1) + I_{k,\Phi}^p(t) - \frac{f}{c} T_k^p(t) \\ + d_{k,\Phi}(t) + d_{k,\Phi}^p(t) + d_\Phi^p(t) + \varepsilon_\Phi\end{aligned} \tag{A.22}$$

The symbols $\Phi_k(t)$ and $\Phi^p(t)$ denote the phase of the receiver generated signal and the phase of the satellite signal respectively, at the epoch t of satellite signal reception. The symbol $N_k^p(1)$ denotes the initial integer ambiguity. The terms $I_{k,\Phi}^p(t)$ and $T_k^p(t)$ are the ionospheric and tropospheric delays. The ionospheric delay factor has a negative value because the carrier phase progresses when travelling through the ionosphere [5]. Furthermore, the tropospheric factor is converted in cycles using the factor f/c, where f is the nominal frequency and c is the speed of light in vacuum. The symbols $d_{k,\Phi}(t)$ and $d_k^p(t)$ refer to the receiver and satellite hardware delays respectively. The symbol $d_{k,\Phi}^p$ denotes the multipath effect and ε_Φ denotes the carrier phase measurement noise [5]. Assuming synchronisation of the satellite and receiver clocks, omitting other error sources (receiver phase tracking circuits, local oscillator, multipath and measurement noise), and taking into account both the time of transmission and reception of the signal, the equation for the phase observation between a satellite i and a receiver A, can be written as follows [13]:

$$\Phi_A^i(\tau) = \Phi^i(t) - \Phi_A(\tau) \tag{A.23}$$

where:

$\Phi_A^i(\tau)$	=	phase reading (phase at receiver A of the signal from satellite i at time τ);
$\Phi^i(t)$	=	received signal (phase of the signal as it left the satellite at time t); and
$\Phi_A(\tau)$	=	generated signal phase (phase of the receiver's signal at time τ).

If $\rho_A^i(t)$ is the range between receiver and satellite, we have:

$$t = \tau - \frac{\rho_A^i(t)}{c} \tag{A.24}$$

Therefore:

$$\Phi^i(t) = \Phi^i\left(\tau - \frac{\rho_A^i(t)}{c}\right) = \Phi^i(\tau) - \underbrace{\frac{\partial \Phi^i(t)}{dt}}_{f} \times \frac{\rho_A^i(t)}{c} + \ldots = \Phi^i(\tau) - f\frac{\rho_A^i(t)}{c} + \ldots \tag{A.25}$$

And finally:

$$\Phi_A^i(\tau) = \Phi^i(\tau) - \frac{f}{c}\rho_A^i(t) - \Phi_A(\tau) + N_A^i \tag{A.26}$$

where:

$\Phi_A^i(\tau)$	=	phase reading (degree or cycles);
$\Phi^i(\tau)$	=	emitted signal;
$\frac{f}{c}\rho_A^i(t)$	=	total number of wavelengths;
$\Phi_A(\tau)$	=	generated signal;
N_A^i	=	integer ambiguity;
f	=	frequency of the carrier; and
c	=	speed of light.

The carrier phase measurement technique typically uses the difference between the carrier phases measured at a reference receiver and a user receiver. Particularly, a double-difference technique is used to remove the satellite and receiver clock errors. This is therefore an inherently differential GPS technique, which is discussed in Chapter 2 of this Thesis.

A.4.3 Doppler Observable

The equation that associates the transmitted frequency from the satellite with the received frequency is:

$$f_k = \frac{f^p}{1+\frac{r'}{c}} \tag{A.27}$$

where f_k is the received frequency, f^p denotes the emitted frequency from the satellite, r' denotes the radial velocity in the satellite-receiver direction and c denotes the speed of light in vacuum. The Doppler frequency shift is given by the difference $f^p - f_k$.

The radial velocity r' is the actual rate of change in the satellite-receiver distance and it is given by:

$$r' = -\frac{f_k - f^p}{f_k}\cdot c \tag{A.28}$$

The integrated Doppler count between two epochs t_1 and t_2 is given by:

$$N_{(t_1,t_2)} = \int_{t_1}^{t_2}\left(f^p - f_k\right)dt \tag{A.29}$$

More information about the Doppler Observable can be found in the literature [4].

A.5 GPS ERROR SOURCES

In this section the most important errors that affect pseudorange and carrier phase measurements are presented in some detail. These errors can be classified into the four broad categories listed below:

- ***Receiver Dependent Errors:***
  - Clock Error; and
  - Noise and Resolution.

- ***Ephemeris Prediction Errors.***
- ***Satellite Dependent Errors:***
  - Clock Offset; and
  - Group Delays.
- ***Propagation Errors:***
  - Ionospheric Delay;
  - Tropospheric Delay; and
  - Multipath.

Moreover, there are other errors to be taken into account, which are related to the ***User Dynamics.*** All of these errors are described in the following paragraphs.

A.5.1 Receiver Clock Error

Most receivers have quartz clocks to measure the GPS time, which are not as accurate as the atomic clocks of the satellites. Therefore, there is an offset between the receiver and satellite clocks called Receiver Clock Error. This error affects both the measurement of the signal flight time and the calculation of the satellite's position at time of transmission.

A.5.2 Receiver Noise and Resolution

The receiver hardware and software design directly affect signal tracking and decoding of the navigation message. Measurement Noise is a random error, which depends entirely on the electronic components of the receiver. Receivers for very precise measurements are designed to minimise this error component. Both noise and resolution errors can be reduced by using appropriate filtering techniques. Theoretically, receiver noise can be removed by averaging the measurements, but only over fairly long periods of observation time.

A.5.3 Ephemeris Prediction Errors

The ephemeris data are required for both pseudorange and phase computations. These errors are due to incorrect estimation of the satellites ephemeris at the MCS. A model for evaluating the errors (Figure A-4) is obtained by considering the three components of the vector representing the difference between estimated and true distance: ATK (along track), XTK (cross track), and RAD (radial).

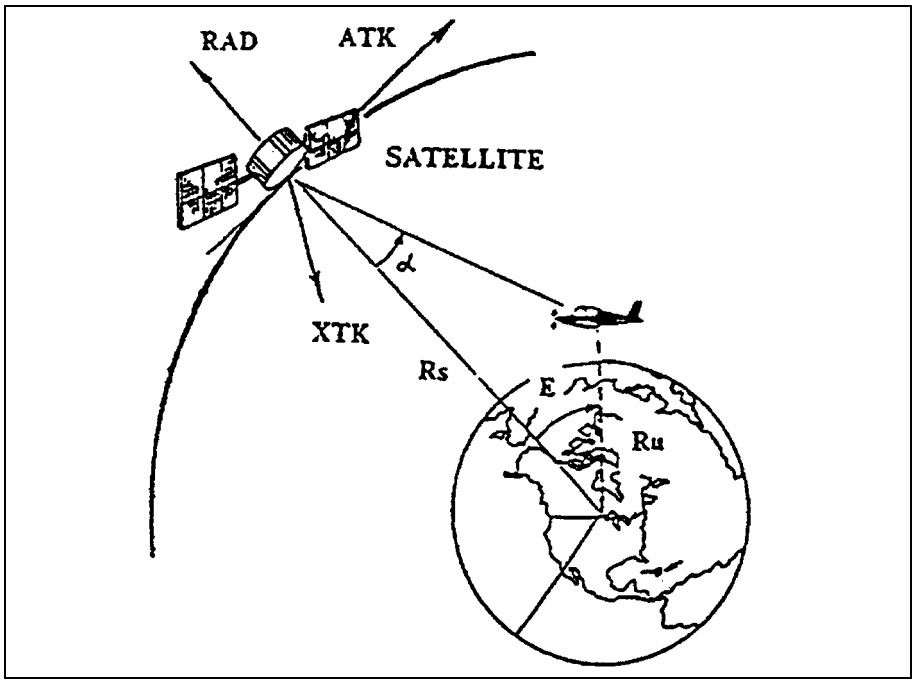

Figure A-4: Error Components in Ephemeris Estimation.

The maximum error is experienced when the satellite has an elevation of 0° on the receiver horizon and the line-of-sight (LOS) user-satellite lies on the geometric plane containing ATK. In general, the error can be expressed as a function of the three components, in the form:

$$ERR = RAD\cos\alpha + ATK\sin\alpha\cos\beta + XTK\sin\alpha\sin\beta \qquad \text{(A.30)}$$

where:

α = angle between the LOS user-satellite and the satellite vertical; and

β = angle between the ATK direction and the plane containing the LOS and the satellite vertical.

Typical values for the error components are in the order of about 1 m along RAD, 7 m along ATK, 3 m along XTK.

The US DOD Precise Ephemeris is calculated from actual observation to the satellites from the monitor stations. It is produced several days after the observation period and is available only to authorised users. Other non-DOD organisations produce precise ephemeris, both globally and locally, by suitable modelling of all forces acting on the satellites. Orbit relaxation techniques can be developed within GPS software. These techniques solve for small orbital errors in the broadcast ephemeris and produce improved relative position. Differential techniques can correct the ephemeris prediction errors (see Chapter 2).

A.5.4 Clock Offset

Corrections to the drift of the satellite atomic clocks are computed by the MCS and then broadcasted to the users in the navigation message. The effect of Satellite Clock Offset is negligible in most positioning applications (using the polynomial coefficients corrections computed at the MCS it is possible to reduce this error down to 1 part per 10^{12}). The residual error is due to the fact that corrections from the MCS are periodic and not continuous. The DOD deliberately manipulated the satellite clocks (see Section A.3) in order to reduce the positional accuracy of non-military receivers.

A.5.5 Group Delays

These are the delays typical of the satellite electronic circuits. They are estimated on the ground before the satellites are launched and corrections are included in the navigation message.

A.5.6 Ionospheric Delay

As the satellite signal passes through the ionosphere, it is delayed for two reasons [14]. Firstly, because it travels through a non-vacuum material (propagation delay); thus the Pseudo Random Noise (PRN) codes are delayed, while the carrier phase is advanced when passing through the ionosphere layers. Secondly, because it bends due to refraction; for many applications, the error caused by the bending effect can be considered negligible if the signals are transmitted by satellites with an elevation of 15° or more.

The ionospheric delay is dependent primarily on the number of electrons that the signal encounters along its propagation path. It is therefore dependent both on the ionosphere characteristics (variable during the day and with seasons), and the path angle (elevation angle of the satellite). It is possible to approximately evaluate the ionospheric delay using the following equation [14]:

$$\Delta\tau = 40.31\frac{TEC}{cf^2} \tag{A.31}$$

where TEC (Total Electron Content) express the electron density (electr./m^2) of an ideal column with section 1 m^2 and the symmetry axis coincident with the propagation path.

The expression for calculating the delay as a function of the elevation angle is the following [14]:

$$\Delta\tau = \frac{0.53Z(El)}{c} \tag{A.32}$$

where *Z(El)* is the obliquity factor.

Depending on the receiver design, different models can be adopted to calculate the correction terms to be applied to the pseudorange before solving the navigation equations. Particularly, C/A code receivers (L1 only), use a sinusoidal model of the ionosphere, derived from statistical studies (Klobuchar model), which take into account the variations of the ionospheric layers (low over night, rapidly getting higher after dawn, getting slightly higher during the afternoon and rapidly getting lower after sunset). The sinusoidal parameters (amplitude and period) are transmitted in the navigation message.

The relevant equations are the following:

$$IDV = DC + A\cos\left[\frac{2\pi(t-\Phi)}{P}\right] \text{ (day)} \tag{A.33}$$

$$IDV = DC \text{ (night)} \tag{A.34}$$

where *IDV* (Ionospheric Vertical Delay) is expressed in nsec, DC is the constant night-day offset (5 nsec), *A* is the amplitude (whose value is between 10 and 100 nsec), Φ is the constant phase offset (14.00 hours), *t* is the local time, and *P* is the period.

The two factors *A* and *P* are transmitted as coefficients of a cubic equation representing a model of the ionosphere with varying latitude. The delay also depends on obliquity of the path and, therefore, elevation is included as an additional factor in the equation:

$$TID = \left[1 + 16(0.53 - El)^3\right] IVD \tag{A.35}$$

where *TVD* is the Total Ionospheric Delay (nsec) and *El* is the elevation angle of the satellites over the horizon. It should also be stated, that the ionosphere will affect the L1 and L2 signals by different values. Therefore, P-code receivers can measure the difference (Δt) between the time of reception of L1 and L2, and evaluate the delay associated with both of them. For L1 we have:

$$\Delta\tau_{L1} = \Delta T\left[\left(\frac{f_{L1}}{f_{L2}}\right)^2 - 1\right]^{-1} \tag{A.36}$$

where:

$$\Delta T = \frac{40.31 \cdot TEC}{cf^2}\left(\frac{1}{f_{L2}^2} - \frac{1}{f_{L1}^2}\right) \tag{A.37}$$

A.5.7 Tropospheric Delay

The troposphere is the lower part of the atmosphere (up to about 50 km) and its characteristics depend on local humidity, temperature and altitude. There are various expressions used to model the tropospheric delay, all including the elevation angle. One of them (not always reliable), is the following:

$$TD = \operatorname{cosec}\left[El\left(1.4588 + 0.0029611 N_S\right)\right] \tag{A.38}$$

where TD is the Tropospheric Delay and N is the superficial refraction factor, dependent on season and latitude. Another model of tropospheric refraction is the following [14]:

$$N = 77.6\frac{P}{T} + 3.73 \times 10^5 \frac{e}{T^2} \tag{A.39}$$

where *P* is the total pressure, *T* is the absolute temperature, and *e* is the partial water vapour pressure.

The refraction of the troposphere can be separated into two components, the dry component ($77.6\,P/T$) and the wet component ($3.73 \times 10^5\, e/T^2$). The dry component is responsible for the 90% of the total tropospheric delay and can be easily modelled. The wet component varies according to the water vapour pressure; therefore it is difficult to be modelled.

Table A-2 shows typical values of the tropospheric delay [14].

Table A-2: Tropospheric Delays

Elevation Angle (°)	Dry Component (m)	Wet Component (m)
90	2.3	0.2
30	4.6	0.4
10	13.0	1.2
5	26.0	2.3

A.5.8 Multipath

GPS signals may arrive at the receiver antenna via different paths, due to reflections by objects along the path. Such effect is known as multipath. The reflected signal will have a different path length compared to the direct signal; therefore it will give a biased distance measurement. Multipath depends on the surrounding environment of the receiver and the satellite geometry. Generally, multipath will be greater for low elevation satellites. For surveying applications, the antenna ground plane prevents the reflected signal from the ground to reach the antenna phase centre [14]. Code multipath is much greater than carrier phase multipath; and the C/A code multipath error is greater than the P code one [15]. Multipath can be modelled only at static points, by taking observations at the same points, at the same hour on consecutive days. This, however, is not practical or even impossible in a dynamic environment. Other interesting techniques use the Signal to Noise Ratio (SNR) to detect and quantify multipath [16].

A.5.9 User Dynamics Errors

There are various errors associated with a dynamic GPS platform. These errors range from the physical masking of the GPS antenna to the accuracy degradation caused by sudden acceleration of the antenna. If carrier phase is used, the resulting effect of "cycle slips" is the need for re-initialisation (i.e., re-determination of the integer ambiguities). In general, a distinction is made between medium-low and high dynamic platforms. Due to the scope of the present dissertation, we will investigate the effect of high dynamic aircraft manoeuvres on GPS performance.

A.6 UERE VECTOR

The User Equivalent Range Error (UERE) is an error vector along the line-of-sight user-satellite given by the projection of all system errors. The value of this vector can be reduced only by careful design of the receiver, since there is nothing the user can do to reduce other error sources. The UERE is generally measured in metres (95%).

The portion of the UERE allocated to the space and control segments is called the User Range Error (URE) and is defined at the phase centre of the satellite antenna. The portion of the UERE allocated to the User Equipment is called the UE Error (UEE). Specifically, the UERE is the root-sum-square of the URE and UEE. Typical values of the UERE vector are in the order of 13 m (95%) for P-code receivers and 15.7 to 23.1 m (95%) for C/A code receivers.

A.7 GPS AND KALMAN FILTERING

The GPS measurement process is corrupted by noise that introduces errors into the calculation. This noise includes the errors in the ionospheric corrections and system dynamics not considered during the measurement process (e.g., user clock drift). A Kalman filter characterises the noise sources in order to minimise their effect on the desired receiver outputs.

The Kalman filter is a linear, recursive estimator that produces the minimum variance estimate in a least square sense under the assumption of white, Gaussian noise processes. There are two basic processes that are modelled by a Kalman filter. The first process is a model describing how the error state vector changes in time (system dynamics model). The second model defines the relationship between the error state vector and any measurements processed by the filter (measurement model).

The Kalman filter sorts out information and weights the relative contributions of the measurements and of the dynamic behaviour of the state vector. The measurements and state vector are weighted by their respective covariance matrices. If the measurements are inaccurate (large variances) when compared to the

state vector estimate, then the filter will de-weight the measurements. On the other hand, if the measurements are very accurate (small variances) when compared to the state vector estimate, then the filter will tend to weight the measurements heavily with the consequence that its previously computed state estimate will contribute little to the latest state estimate.

When the GPS receiver is aided or integrated with other navigation sensors (e.g., an Inertial Navigation System, an external clock or an altimeter), then the Kalman filter can be extended to include the measurements added by these sensors. In fact, a typical implementation for integrated systems would be to have a central Kalman filter incorporating measurements from all available sources. More detailed information about Kalman filters for GPS receivers and GPS-based integrated navigation systems can be found in the literature [9].

A.8 GPS MODERNIZATION

Current GPS Modernization efforts focus on improving position and timing accuracy, availability, integrity monitoring support capability and enhancement to the control system. As these system enhancements are introduced, users will be able to continue to use existing receivers, as signal backward compatibility is an absolute requirement for both the military and civil user communities. Although current GPS users will be able to operate at the same, or better, levels of performance that they enjoy today, users will need to modify existing user equipment or procure new user equipment in order to take full advantage of any new signal structure enhancements [17].

GPS modernization is a multi-phase effort to be executed over the next 15 years. Additional signals are planned to enhance the ability of GPS to support civil users and provide a new military code. The first new signal will be the C/A code on the L2 frequency (1227.60 MHz). This feature will enable dual channel civil receivers to correct for ionospheric error. A third civil signal will be added on the L5 frequency (1176.45 MHz) for use in safety-of-life applications. L5 can serves as a redundant signal to the GPS L1 frequency (1575.42 MHz) with a goal of assurance of continuity of service potentially to provide precisions approach capability for aviation users. In addition, a secure and spectrally separated Military Code (M-Code) will be broadcast on the L1 and L2 frequencies enabling the next generation of military receivers to operate more fully in an electronic jamming environment. At least one satellite is planned to be operational on orbit with the new C/A on L2 and M-Code capability no later than 2003. The Initial Operating Capability (IOC) with 18 satellites on orbit is planned for 2008 and the Full Operational Capability (FOC) with 24 satellites on orbit is planned for 2010. At least one satellite is planned to be operational on orbit with the new L5 capability no later than 2005, with IOC planned for 2012 and FOC planned for 2014 [18].

A.9 REFERENCES

[1] Parkinson, B.W. and Spilker, J.J., Jr. Editors. (1996). “Global Positioning System: Theory and Applications – Volumes I and II”. Progress in Astronautics and Aeronautics. Vol. 163. Published by the American Institute of Aeronautics and Astronautics (AIAA).

[2] Kayton, M. and Fried, W.R. (1997). “Avionics Navigation Systems”. Second Edition. Wiley Interscience Publication.

[3] Kaplan, E.D. (Editor). (1996). “Understanding GPS Principles and Applications”. Artech House Publishers, New York (USA).

[4] Seeber, G. (1994). “Satellite Geodesy”. Second Edition. Artech House Publishers, New York (USA).

[5] Leick, A. (1995). "GPS Satellite Surveying". Second Edition. Department of Surveying Engineering – University of Maine (USA).

[6] AGARD LS-161. (1990). "The NAVSTAR GPS System". NATO Advisory Group for Aerospace Research and Development. Neuilly-sur-Seine (France).

[7] ASHTECH Inc. (1996). "ASHTECH Z-12 Receiver". Technical Specification Leaflet.

[8] Thomis, D.M. (1984). "Global Positioning System – A Modification to the Baseline Satellite Constellation for Improved Geometric Performance". US Air Force Institute of Technology.

[9] Siouris, G.M. (1993). "Aerospace Avionics Systems". Academic Press, San Diego, California (USA).

[10] Maybeck, P.S. (1979). "Stochastic Models, Estimation, and Control". Vol. I, Academic Press, New York (USA).

[11] Philips, A.H. (1990). "The Determination of PDOP in GPS". AGARD-AG-314. NATO Advisory Group for Aerospace Research and Development. Neuilly-sur-Seine (France).

[12] Vagias, C. (1995). "High Precision Differential GPS Using New Hardware". MSc Dissertation. Institute of Engineering Surveying and Space Geodesy (IESSG) – University of Nottingham.

[13] Ashkenazi, V. (1997). "Principles of GPS and Observables". MSc Lecture Notes. Institute of Engineering Surveying and Space Geodesy (IESSG) – University of Nottingham.

[14] Dodson, A.H. (2002). "Propagation Effects on GPS Measurements". MSc Lecture Notes. Institute of Engineering Surveying and Space Geodesy (IESSG) – University of Nottingham.

[15] Walsh, D. (1994). "Kinematic GPS Ambiguity Resolution". PhD Thesis. Institute of Engineering Surveying and Space Geodesy (IESSG) – University of Nottingham.

[16] Adham, T.A. (1995). "Multipath Effects on Carrier Phase". MSc Dissertation. Institute of Engineering Surveying and Space Geodesy (IESSG) – University of Nottingham.

[17] Dye, S. and Baylin, F. (2004). "The GPS Manual Principles and Applications". Second Edition. Baylin Publications (USA).

[18] US Department of Defense and US Department of Transportation. (2001). "Federal Radionavigation Plan – 2001". National Technical Information Services. Springfield (VA). Doc. DOT-VNTSC-RSPA-01-3/DOD-4650.5.