CHAPTER 3

Equations for Unsteady Motions in Combustion Chambers

The examples described in Chapter 1, and many others, establish a firm basis for interpreting unsteady motions in a combustor in terms of acoustic modes of the chamber. That view has been formalized during the past fifty years and has led to the most widely used methods for interpreting combustor dynamics. In this and the following chapter, we present the foundations of a particularly successful version of methods based on expansion in normal acoustic modes and spatial averaging. We assume familiarity with the basic ideas of classical fluid dynamics and acoustics. Chapter 5 covers the principles and chief results of classical acoustics required as part of the foundation for understanding combustion instabilities. The discussions in Chapters 3 and 4 are quite formal, intended to serve as the basis for a framework within which unsteady motions, especially combustion instabilities, may be treated for all types of combustors. Hence the physical model for which the formalism is developed is quite general.

3.1. Modes of Wave Motion in a Compressible Medium

In this section, the term 'modes' refers not to natural motions or resonances of a chamber but means rather a *type* or class of motions in compressible flows generally. The brief discussion here is intended to address the question: How is it possible that apparently coherent, nearly classical acoustic waves exist in chambers containing highly turbulent non-uniform flow? It's a fundamentally important observation that such is the case. The explanation has been most thoroughly clarified by Chu and Kovasznay (1957), who combined and elaborated some results known for nearly a century. Their conclusions most significant for present purposes may be summarized as follows:

(i) Any small amplitude (linear) disturbance may be synthesized of three modes of propagation: *entropy waves* or 'spots', small regions having temperatures slightly different from the ambient temperature of the flow; *vortical or shear waves* characterized by nonuniform vorticity; and *acoustic waves.*

(ii) In the linear approximation, *if the flow is uniform,* the three types of waves propagate independently, but may be coupled at boundaries (e.g. nozzles) or in combustion zones.

Entropy and vortical waves having small amplitude propagate (are 'convected') in a uniform field with the mean flow speed, but acoustic waves propagate with their own speeds of sound. Moreover, in the linear limit, only acoustic waves carry disturbances of pressure. All three types of waves possess velocity fluctuations. If the medium is non-uniform or the unsteady motions have finite amplitudes, the three modes become coupled. As a result, each of the waves may then carry pressure, temperature and velocity fluctuations. Little extension of the fundamental theory has been accomplished (see Chu and Kovasznay) and what understanding exists has been gained from considerations of particular problems. Some of the consequences of these types of modal coupling arise in the theory developed here, but much remains to be investigated.

Long experience has established the wide applicability of the basic physical model of combustion instabilities as acoustic waves propagating in a non-uniform flow. Vorticity and entropy waves accompany turbulence in a combustor but may also have other origins, such as flow separation, the unsteady behavior

of injection devices, and interactions of the acoustic field with other processes, notably flow injected at the lateral boundaries. Consequently, both the average and time-varying velocity fields in a combustor comprise irrotational and rotational motions. Observational evidence for combustion instabilities suggests that the rotational motions are in some sense often secondary, initially one of the chief guiding principles for the formalism developed in this chapter, but they cannot be ignored. In fact, both steady and unsteady rotational motions participate significantly in combustion instabilities in solid propellant rockets. Several examples of the generation and behavior of vorticity have been intensively studied in the past ten years. We will discuss some of the results in Chapter 6.

The main idea at this point is that in lowest approximation the unsteady field can be expressed as a synthesis of classical acoustic modes having time-varying amplitudes. Then the purpose of the analysis is to work out a means for computing the changes of those amplitudes due to various perturbations. Departures from the simplest classical acoustic behavior arise from the actions of the other physical and chemical processes taking place in a combustion chamber. All of those processes, except basic gasdynamics, are assumed to have relatively weak effects, producing small shifts of the classical acoustic frequencies and, more significantly, small fractional changes of the modal amplitudes during a period of oscillation. Hence there are two small quantities naturally characterizing the procedure: A reference Mach number for the unsteady velocity is a measure of the acoustic amplitudes; and a typical Mach number of the average flow measures perturbations of the classical acoustic behavior. Much of this chapter is concerned with reduction of the general equations of motion by expansion in those two small parameters. In applications of the formalism and interpretation of the results, it is essential to understand and maintain the distinction between the roles of the two parameters. Failure to do so leads to confusion and false conclusions.

Despite the emphasis on the acoustic field, this procedure does *not* exclude the existence of rotational motions. In the expansion procedure they arise from inhomogeneous terms in the equations for the higher order terms. This fundamental point has been missed by several workers in this field and has led at least to misunderstandings and occasionally to misleading or incorrect analyses. The origins of the difficulties will be clarified in later discussions, particularly in Chapter 6, but in view of the considerable confusion about the matter, it is important to begin addressing the matter here. It is not a new idea. Flandro (1967) in his Ph.D. thesis first used the general expansion procedure developed in this chapter to work out a problem involving interactions between acoustic and vorticity fields leading to roll torques in solid propellant rockets. He and others have subsequently investigated other examples of the influences of vorticity on acoustic waves using this approach, occasionally with controversial and sometimes incorrect results. (See, for example, Flandro 1995; Culick *et al.* 1991; Culick 1998; Swenson and Culick 1998; Seywert and Culick, 1998; Flandro and Malhotra 1995; Malhotra, Flandro and Roh 2000; Malhotra and Flandro 2001, 2002; Majdalani and Van Moorhem 1998; Majdalani 2004; Flandro and Majdalani 2003.)

3.2. Equations of Motion in a Reacting Flow

Combustion systems commonly contain condensed phases: liquid fuel or oxidizer, and combustion products including soot and condensed metal oxides. Hence the equations of motion must be written for two or three phases consisting of at least one species each. For investigating the dynamics of combustors, it seems entirely adequate to consider two phases (gas and a condensed phase comprising both liquid and solid particles). The properties of each phase are represented as mass averages of the properties of all member species. For a medium consisting of a multicomponent mixture of reacting gases and, for simplicity, a single condensed phase, it is a straightforward matter to construct a system of equations representing a single fluid. The procedure is explained in Annex A. As a result we can treat combustor dynamics under broad conditions as unsteady motions of a fluid having the mass-averaged properties of the actual medium.[1] The dimensional governing equations are (A.59)–(A.64):

[1] We now use $C_v, \gamma, R, \ldots$ to stand for the mass-averaged properties represented by boldface symbols in Annex A.

Conservation of Mass:
$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} + \mathcal{W} \tag{3.1}$$

Conservation of Momentum:
$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p + \boldsymbol{\mathcal{F}} \tag{3.2}$$

Conservation of Energy:
$$\rho C_v \left[\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right] = -p \nabla \cdot \mathbf{u} + \mathcal{Q} \tag{3.3}$$

Equation for the Pressure:
$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u} + \mathcal{P} \tag{3.4}$$

Equation for the Entropy:
$$\rho \frac{Ds}{Dt} = \frac{1}{T} \mathcal{S} \tag{3.5}$$

Equation of State:
$$p = R \rho T \tag{3.6}$$

Definitions of all symbols are given in Annex A.

It is particularly important to realize that the source functions $\mathcal{W}$, $\boldsymbol{\mathcal{F}}$, $\mathcal{Q}$, $\mathcal{P}$ and $\mathcal{S}$ in principle contain all relevant processes in the systems to be analyzed here. They include, for example, the modeling and representations of the actions of actuation mechanisms used for active control. Eventually, the most difficult problems arising in this field are associated with modeling the physical processes dominant in the problems addressed.

For both theoretical and computational purposes it is best to express the equations in dimensionless variables using the reference values:

L : reference length

ρ_r, p_r, T_r, a_r : reference density, pressure, temperature and speed of sound

C_{vr}, C_{pr}, R_r : reference values of C_v, C_p, R

Then define the dimensionless variables represented by $\mathbf{M}$, and *for simplicity use the same symbols used for dimensional variables*:

$$\mathbf{M} = \frac{\mathbf{u}}{a_r}; \qquad \frac{\rho}{\rho_r} \to \rho; \qquad \frac{p}{\rho_r a_r^2} \to p; \qquad \frac{T}{T_r} \to T; \qquad \frac{C_v}{C_{pr}} \to C_v; \qquad \frac{C_p}{C_{pr}} \to C_p;$$
$$\frac{R}{C_{pr}} \to R; \qquad \frac{a_r}{L} t \to t; \qquad \frac{s}{C_{pr}} \to s$$

The dimensionless source functions are

$$\frac{L}{\rho_r a_r} \mathcal{W} \to \mathcal{W}; \qquad \frac{L}{\rho_r a_r^2} \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}}; \qquad \frac{L}{\rho_r a_r^3} \mathcal{Q} \to \mathcal{Q}; \qquad \frac{L}{\rho_r a_r} \mathcal{P} \to \mathcal{P}; \qquad \frac{\mathcal{S}}{\rho_r a_r C_{vr}} \to \mathcal{S}$$

For consistent definitions, $p_r = \rho_r R_r T_r$ and $R_r = C_{pr} = C_{vr}$, so in dimensionless form, the relations $R = C_p - C_v$ and $\gamma = C_p / C_v$ still hold.

Substitution of these definitions in equations (3.1)–(3.6) leads to the set of dimensionless equations for the single fluid model:

Mass:
$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{M} + \mathcal{W} \tag{3.7}$$

Momentum:
$$\rho \frac{D\mathbf{M}}{Dt} = -\nabla p + \boldsymbol{\mathcal{F}} \tag{3.8}$$

$$\textit{Energy:} \qquad \rho C_v \frac{DT}{Dt} = -p\nabla \cdot \mathbf{M} + Q \tag{3.9}$$

$$\textit{Pressure:} \qquad \frac{Dp}{Dt} = -\gamma p \nabla \cdot \mathbf{M} + \mathcal{P} \tag{3.10}$$

$$\textit{Entropy:} \qquad \rho \frac{Ds}{Dt} = \frac{1}{T}\mathcal{S} \tag{3.11}$$

$$\textit{State:} \qquad p = \rho R T \tag{3.12}$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{M} \cdot \nabla \tag{3.13}$$

We emphasize again that the source terms accommodate all relevant physical processes and can be interpreted to include the influences of actuation used in active control.

3.3. Two-Parameter Expansion of the Equations of Motion

The general equations (3.7)–(3.12) are written in the form suggestive of problems that are dominated by fluid mechanical processes, a tactic dictated by the observations described earlier. This point of view is the basis for the approach taken here to construct a general framework within which both practical and theoretical results can be obtained by following systematic procedures.

We are not concerned at this point with simulations or other methods relying essentially on some sort of numerical analysis and large scale computations. The nature of the problems we face suggests perturbation methods of solution. If the source terms $\mathcal{W}$, ... were absent from (3.7)–(3.11), the homogeneous equations then represent nonlinear inviscid motions in a compressible fluid: Nonlinear acoustics in a medium without losses. One useful method for investigating such problems is based on expansion of the equations in a small parameter, ε, measuring the amplitude of the motion. Specifically, ε can be taken equal to M_r', a Mach number characteristic of the fluctuating flow, $\varepsilon = M_r'$.

The problems we are concerned with here are defined essentially by the non-zero functions $\mathcal{W}$, ... Because observed behavior seems to be dominated by features recognizable as 'acoustical', those sources which excite and sustain the actual motions must in some sense be small. They should therefore be characterized by at least one additional small parameter. It has become customary to select only one such parameter, $\mu = \bar{M}_r$, a Mach number $\bar{M}_r$ characterizing the mean flow, for the following reasons.[2]

Any operating combustion chamber contains an average steady flow produced by combustion of the fuel and oxidizer to generate products. The intensity of the flow, partly measurable by the Mach number, is therefore related to the intensity of combustion; both processes can in some sense be characterized by the same quantity, namely the Mach number of the average flow. Thus many of the processes represented in the source functions may be characterized by μ, in the sense that their influences become vanishingly small as $\mu \to 0$ and are absent when $\mu = 0$.

It is important to understand that the two small parameters ε and μ have different physical origins. Consequently, they also participate differently in the formal perturbation procedures. Familiar nonlinear gas dynamical behavior is, in the present context, governed by the parameter ε; steepening of compressive waves

[2] We will use the symbols ε and μ rather than M_r' and $\bar{M}_r$ to simplify writing, and to emphasize the special positions held by the two independent sorts of perturbations.

is a notable example. In the expansion procedure worked out here, the term 'nonlinear behavior' refers to the consequences of terms higher order in ε.

On the other hand, the parameter μ characterizes perturbations of the gasdynamics due in the first instance to combustion processes and the mean flow. Terms of higher order in μ, but linear in ε, represent linear processes in this scheme. Failure to recognize this basic distinction between ε and μ can lead to incorrect applications of formal procedures such as the method of time-averaging. Instances of this point will arise as the analysis is developed.

3.3.1. Expansion in Mean and Fluctuating Values. There is no unique procedure for carrying out a two-parameter expansion. We begin here by writing all dependent variables as sums of mean $\bar{(\ )}$ and fluctuating $(\ )'$ parts without regard to ordering:

$$p = \bar{p} + p', \quad \mathbf{M} = \bar{\mathbf{M}} + \mathbf{M}', \ldots, \quad \mathcal{W} = \bar{\mathcal{W}}', \quad \boldsymbol{\mathcal{F}} = \bar{\boldsymbol{\mathcal{F}}} + \boldsymbol{\mathcal{F}}', \ldots \tag{3.14}$$

We take the fluctuations of the primary flow variables (p', $\mathbf{M}'$, ρ', T', s') to be all of the same order in the amplitude ε of the unsteady motion. Generally, the source terms are complicated functions of the flow variables and therefore their fluctuations will contain terms of many orders in ε. For example, suppose $\mathcal{W} = kp^3$. Then setting $p = \bar{p} + p'$ and expanding, we have

$$\mathcal{W} = k(\bar{p} + p')^3 = k\left[\bar{p}^3 + 3\bar{p}^2 p' + 3\bar{p}p'^2 + p'^3\right]$$

Hence we define orders of the fluctuations of the source $\mathcal{W}$ and write

$$\mathcal{W} = \bar{\mathcal{W}} + \mathcal{W}_1' + \mathcal{W}_2' + \mathcal{W}_3' + \mathcal{W}_4' + \ldots$$

where the subscript denotes the order with respect to the amplitude: Here, for the example, $\mathcal{W} = kp^3$ and $\mathcal{W}_2' = (3k\bar{p})p'^2$. All source functions are expressed in this general fashion, but modeling is required to give explicit formulas.

Most combustors contain flows of relatively low Mach number, say $|\bar{\mathbf{M}}| \lesssim 0.3$ or so. Thus we can assume that for a broad range of circumstances, processes depending on the square of $\bar{\mathbf{M}}$, i.e. of order μ^2, probably have small influences on the unsteady motions. We therefore neglect all terms of order μ^2 and higher in the equations. As a practical matter, the equations are greatly simplified with this assumption which we adopt throughout this work.

After substituting all variables split into sums of mean and fluctuating values, and collection of terms by orders, we can rewrite (3.7)–(3.13) as

$$\begin{aligned}&\left[\frac{\bar{D}\bar{\rho}}{Dt} + \bar{\rho}\nabla\cdot\bar{\mathbf{M}} + \bar{\mathbf{M}}\cdot\nabla\bar{\rho} - \bar{\mathcal{W}}\right] + \left[\frac{\partial\rho'}{\partial t} + \bar{\rho}\nabla\cdot\mathbf{M}'\right]\\ &+ \left[\bar{\mathbf{M}}\cdot\nabla\rho' + \rho'\nabla\cdot\bar{\mathbf{M}} + \mathbf{M}'\cdot\nabla\bar{\rho} + \nabla\cdot(\rho'\mathbf{M}')\right] - \mathcal{W}' = 0\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\left[\bar{\rho}\frac{\bar{D}\mathbf{M}'}{Dt} + \nabla\bar{p} - \bar{\boldsymbol{\mathcal{F}}}\right] + \left[\bar{\rho}\frac{\partial\bar{\mathbf{M}}}{\partial t} + \nabla p'\right] + \left[\bar{\rho}\left(\bar{\mathbf{M}}\cdot\nabla\mathbf{M}' + \mathbf{M}'\cdot\nabla\bar{\mathbf{M}}\right) + \rho'\frac{\bar{D}\bar{\mathbf{M}}}{Dt}\right]\\ &+ \left[\rho'\frac{\partial\mathbf{M}'}{\partial t} + \bar{\rho}\mathbf{M}'\cdot\nabla\mathbf{M}' + \rho'\left(\bar{\mathbf{M}}\cdot\nabla\mathbf{M}' + \mathbf{M}'\cdot\nabla\bar{\mathbf{M}}\right)\right] + \left[\rho'\mathbf{M}'\cdot\nabla\mathbf{M}'\right] - \boldsymbol{\mathcal{F}}' = 0\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\left[\bar{\rho}C_v\frac{\bar{D}\bar{T}}{Dt} + \bar{p}\nabla\cdot\bar{\mathbf{M}} - \bar{\mathcal{Q}}\right] + C_v\left[\bar{\rho}\frac{\partial T'}{\partial t} + \bar{p}\nabla\cdot\mathbf{M}'\right] + \left[\bar{\rho}C_v\left(\bar{\mathbf{M}}\cdot\nabla T' + \mathbf{M}'\cdot\nabla\bar{T}\right) + C_v\rho'\frac{\bar{D}\bar{T}}{Dt} + p'\nabla\cdot\bar{\mathbf{M}}\right]\\ &+ \left[C_v\bar{\rho}\frac{\partial T'}{\partial t} + C_v\rho'\left(\bar{\mathbf{M}}\cdot\nabla T' + \mathbf{M}'\cdot\nabla\bar{T}\right) + C_v\rho'\mathbf{M}'\cdot\nabla T' + p'\nabla\cdot\mathbf{M}'\right] + \left[C_v\bar{\rho}\mathbf{M}'\cdot\nabla T'\right] - \mathcal{Q}' = 0\end{aligned} \tag{3.17}$$

$$\left[\frac{\bar{D}\bar{p}}{Dt}+\gamma\bar{p}\nabla\cdot\bar{\mathbf{M}}-\bar{p}+\gamma\bar{p}\nabla\cdot\bar{\mathbf{M}}-\bar{\mathcal{P}}\right]+\left[\frac{\partial p'}{\partial t}+\gamma\bar{p}\nabla\cdot\mathbf{M}'\right]+\left[\bar{\mathbf{M}}\cdot\nabla p'+\mathbf{M}'\cdot\nabla\bar{p}+\gamma p'\nabla\cdot\bar{M}\right]$$
$$+\left[\mathbf{M}'\cdot\nabla p'+\gamma p'\nabla\cdot\mathbf{M}'\right]-\mathcal{P}'=0 \tag{3.18}$$

$$\left[\bar{\rho}\bar{T}\frac{\bar{D}\bar{s}}{Dt}-\bar{\mathcal{S}}\right]+\left[\bar{\rho}\bar{T}\frac{\partial s'}{\partial t}\right]+\left[\bar{\rho}\bar{\mathbf{M}}\cdot\nabla s'+\rho'\bar{T}\frac{\bar{D}\bar{s}}{Dt}+\bar{\rho}\bar{T}\mathbf{M}'\cdot\nabla\bar{s}+\bar{\rho}T'\bar{\mathbf{M}}\cdot\nabla\bar{s}\right]$$
$$+\left[\rho'\bar{T}\frac{\partial s'}{\partial t}+\rho'T'\frac{\partial D\bar{s}}{\partial t}+\rho'\bar{T}\mathbf{M}'\cdot\nabla\bar{s}+\bar{\rho}T'\mathbf{M}'\cdot\nabla\bar{s}+\bar{\rho}T'\frac{\partial s'}{\partial t}+\bar{\rho}\bar{T}\mathbf{M}'\cdot\nabla\mathcal{S}'\right] \tag{3.19}$$
$$+\left[\rho'T'\frac{\partial s'}{\partial t}+\left(\bar{\rho}T'+\rho'\bar{T}\right)\mathbf{M}'\cdot\nabla s'+\rho'T'\left(\mathbf{M}'\cdot\nabla\bar{s}+\bar{\mathbf{M}}\cdot\nabla s'\right)\right]+\left[\rho'T'\mathbf{M}'\cdot\nabla s'\right]-\mathcal{S}'=0$$

$$\left[\bar{p}-R\bar{\rho}\bar{T}\right]+\left[p'-R\left(\bar{\rho}T'+\rho'\bar{T}\right)\right]+\left[-R\rho'T'\right]=0 \tag{3.20}$$

where the convective derivative following the mean flow is

$$\frac{\bar{D}}{Dt}=\frac{\partial}{\partial t}+\bar{\mathbf{M}}\cdot\nabla \tag{3.21}$$

As a convenience in writing, it is useful to introduce some symbols defining groups of ordered terms. The set of equations (3.15)–(3.20) then become:

$$\left[\frac{\bar{D}\bar{\rho}}{Dt}+\bar{\rho}\nabla\cdot\bar{\mathbf{M}}-\bar{\mathcal{W}}\right]+\left(\frac{\partial\rho'}{\partial t}+\bar{\rho}\nabla\cdot\mathbf{M}'\right)+\{[\rho]\}_1+\{\rho\}_2-\mathcal{W}'=0 \tag{3.22}$$

$$\left[\bar{\rho}\frac{\bar{D}\bar{\mathbf{M}}}{Dt}+\nabla\bar{p}-\bar{\boldsymbol{\mathcal{F}}}\right]+\left(\bar{\rho}\frac{\partial\mathbf{M}'}{\partial t}+\nabla p'\right)+\{[\mathbf{M}]\}_1+\{\mathbf{M}\}_2+\{\mathbf{M}\}_3+\{[\mathbf{M}]\}_2-\boldsymbol{\mathcal{F}}'=0 \tag{3.23}$$

$$\left[\bar{\rho}C_v\frac{\bar{D}\bar{T}}{Dt}+\bar{p}\nabla\cdot\bar{\mathbf{M}}-\bar{\mathcal{Q}}\right]+C_v\left(\bar{\rho}\frac{\partial T'}{\partial t}+\bar{p}\nabla\cdot M'\right)+\{[T]\}_1+\{T\}_2+\{T\}_3+\{[T]\}_2-\mathcal{Q}'=0 \tag{3.24}$$

$$\left[\frac{\bar{D}\bar{p}}{Dt}+\gamma\bar{p}\nabla\cdot\bar{\mathbf{M}}-\bar{\mathcal{P}}\right]+\left(\bar{\rho}C_v\frac{\partial P'}{\partial t}+\bar{p}\nabla\cdot M'\right)+\{[p]\}_1+\{p\}_2-\mathcal{P}'=0 \tag{3.25}$$

$$\left[\bar{\rho}\bar{T}\frac{\bar{D}\bar{s}}{Dt}-\bar{\mathcal{S}}\right]+\left(\bar{\rho}\bar{T}\frac{\partial s'}{\partial t}\right)+\{[s]\}_1+\{s\}_2+\{s\}_3+\{[s]\}_2+\{s\}_4-\mathcal{S}'=0 \tag{3.26}$$

$$\left[\bar{p}-R\bar{\rho}\bar{T}\right]+\{p-R\rho T\}_1+\{R\rho T\}_2=0 \tag{3.27}$$

The definitions of the bracketed terms $\{\rho\}_1$, $\cdots$ etc. are given in Annex A, Section A.6; the subscript $\{\quad\}_n$ on the brackets identifies the orders of terms with respect to the fluctuations of flow variables, and the square brackets $[\quad]$ indicate that the terms are first order in the average Mach number. We have shown here in each equation terms of the highest order fluctuations generated by the purely fluid mechanical contributions plus sources that must be expanded to orders appropriate to particular applications. Only the entropy equation produces terms of fourth order.

Time derivatives of quantities identified with the mean flow are retained to accommodate variations on a time scale long relative to the scale of the fluctuations. This generality is not normally required for treating combustion instabilities and unless otherwise stated, we will assume that all averaged quantities are independent of time.

3.3.2. Equations for the Mean Flow. At this point we have two choices. Commonly the assumption is made that the variables of the mean flow 'satisfy their own equations'. That implies that the brackets [ ] vanish identically. With the time derivatives absent, the equations for the mean flow are:

$$\bar{\mathbf{M}} \cdot \nabla \bar{\rho} + \bar{\rho} \nabla \cdot \bar{\mathbf{M}} = \bar{\mathcal{W}} \tag{3.28}$$

$$\bar{\rho} \bar{\mathbf{M}} \cdot \nabla \bar{\mathbf{M}} + \nabla \bar{p} = \boldsymbol{\mathcal{F}} \tag{3.29}$$

$$\bar{\rho} C_v \bar{\mathbf{M}} \cdot \nabla \bar{T} + \bar{p} \nabla \cdot \bar{\mathbf{M}} = \bar{\mathcal{Q}} \tag{3.30}$$

$$\bar{\mathbf{M}} \cdot \nabla \bar{p} + \gamma \bar{p} \nabla \cdot \bar{\mathbf{M}} = \bar{\mathcal{P}} \tag{3.31}$$

$$\bar{\rho} \bar{T} \bar{\mathbf{M}} \cdot \nabla \bar{s} = \bar{\mathcal{S}} \tag{3.32}$$

$$\bar{p} = R \bar{\rho} \bar{T} \tag{3.33}$$

This set of equations certainly applies when the average flow is strictly independent of time and there are no fluctuations. The time derivatives cannot be ignored when the flow variables change so slowly that the motion may be considered as 'quasi-steady' and fluctuations are still ignorable.

It is possible that when fluctuations are present, interactions among the flow variables cause transfer of mass, momentum and energy between the fluctuating and mean flows, generating time variations of the averaged variables. Then the appropriate equations are obtained by time-averaging (3.22)–(3.27) to give[3]

$$\frac{\bar{D}\bar{\rho}}{Dt} + \bar{\rho} \nabla \cdot \bar{\mathbf{M}} = \bar{\mathcal{W}} - \overline{\{[\rho]\}}_1 - \overline{\{\rho\}}_2 + \overline{\mathcal{W}}' \tag{3.34}$$

$$\bar{\rho} \frac{\bar{D}\bar{\mathbf{M}}}{Dt} + \nabla \bar{p} = \bar{\boldsymbol{\mathcal{F}}} - \overline{\{[\mathbf{M}]\}}_1 - \overline{\{\mathbf{M}\}}_2 - \overline{\{\mathbf{M}\}}_3 - \overline{\{[\mathbf{M}]\}}_2 + \overline{\boldsymbol{\mathcal{F}}'} \tag{3.35}$$

$$\bar{\rho} C_v \frac{\bar{D}\bar{T}}{Dt} + \bar{p} \nabla \cdot \bar{\mathbf{M}} = \bar{\mathcal{Q}} - \overline{\{[T]\}}_1 - \overline{\{T\}}_2 - \overline{\{T\}}_3 + \bar{\mathcal{Q}}' \tag{3.36}$$

$$\frac{\bar{D}\bar{p}}{Dt} + \gamma \bar{p} \nabla \cdot \bar{\mathbf{M}} = \bar{\mathcal{P}} - \overline{\{p\}}_1 - \overline{\{p\}}_2 + \bar{\mathcal{P}}' \tag{3.37}$$

$$\bar{\rho} \bar{T} \frac{\bar{D}\bar{s}}{Dt} = \bar{\mathcal{S}} - \overline{\{s\}}_1 - \overline{\{s\}}_2 - \overline{\{s\}}_3 - \overline{\{[s]\}}_2 - \overline{\{s\}}_4 + \bar{\mathcal{S}}' \tag{3.38}$$

$$\bar{p} = \mathbf{R} \bar{\rho} \bar{T} - \overline{\{\rho T\}}_1 - \overline{\{\rho T\}}_2 \tag{3.39}$$

[3]Note that the fluctuations of the source terms denoted by $\mathcal{W}' \cdots$ etc., actually contain squares and higher order products of the dependent variables; hence their time averages will generally be non-zero.

If the mean flow is strictly independent of time, then time averages of all first-order brackets, $\overline{\{\quad\}}_1$, must vanish. For generality we allow them to be nonzero. There seem to be no analyses in which their variations have been taken into account.

The two sets of equations governing the mean flow in the presence of unsteady motion define two distinct formulations of the general problem. In the first, equations (3.28)–(3.33), computation of the mean flow is uncoupled from that of the unsteady flow. Hence formally we are concerned with the stability and time evolution of disturbances superposed on a given, presumed known, mean flow unaffected by the unsteady motions. That is the setting for all investigations of combustion instabilities founded on the splitting of small flow variables into sums of mean and fluctuating values. This approach excludes, for example, possible influences of oscillations on the mean pressure in the chamber (often called 'DC shift'), not an unusual occurrence in solid propellant rockets. When they occur, DC shifts of this sort are almost always unacceptable in operational motors. They may be directly affected by the fluctuations, or they may be largely due to changes in the mean burning rate.

In contrast, the set (3.34)–(3.39) is strongly coupled to the fluctuating field. The situation is formally that producing the problem of 'closure' in the theory of turbulent flows (see, for example, Tennekes and Lumley, 1972). We will not explore the matter here, but note only that the process of time averaging terms on the right hand sides of the equations introduces functions of the fluctuations that are additional unknowns. Formal analysis then requires that those functions be modeled; perhaps the most familiar example in the theory of turbulence is the introduction of a 'mixing length' as part of the representation of stresses associated with turbulent motions. The set (3.34)–(3.39) also can be used to compute 'DC shifts' for specified fluctuations of the flow variables. No results have been reported.

Numerical simulations of combustion instabilities do not exhibit the problem of closure if the complete equations are used, avoiding the consequences of the assumption (3.14). Thus, for example, the results obtained by Levine and Baum (1982, 1988) do show time-dependence of the average pressure in examples of instabilities in solid rockets. Another possible cause of that behavior, probably more important in many cases, is nonlinear dependence of the burning rate on the pressure or velocity near the surface of a solid propellant rocket. Within the structure given here, that behavior arises from time-averaged functions of p', $\mathbf{M}'$, ... contained in the boundary conditions.

We use in this book the formulation assuming complete knowledge of the mean flow, given either by suitable modeling or by solution to the governing equations (3.28)–(3.33). This may in some cases be an important omission. Probably the most important consequence is that the effect of oscillations on burning rate and mean pressure in solid propellant rockets is not covered. See Section 2.1, for example Figure 2.6. This is often, especially in tactical rockets, a significant matter which has been treated by the author only with preliminary calculations in the present scheme. Flandro (private communication) is currently working on this problem.

3.3.3. Systems of Equations for the Fluctuations. The general equations of motion (3.22)–(3.27) and those for the mean flow written in Section 3.3.1 contain a restriction only on the magnitude of the average Mach number. Such generality blocks progress with the analysis and for many applications is unnecessary. The set of equations (3.22)–(3.27) must be simplified to forms that can be solved to give useful results. Many possibilities exist. We follow here a course that previous experience has shown to be particularly fruitful for investigations of combustor dynamics. The choices of approximations and tactics are usually motivated by eventual applications and the type of analysis used.

First we assume that the mean flow is determined by its own system of equations; that is, we avoid the problem of closure and use the first formulation, equations (3.28)–(3.33), discussed in Section 3.3.1. Consequently, the mean flow is taken to be independent of time and the combinations in square brackets [ ], equations (3.22)–(3.27), vanish identically; we write the equations in the form

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla \cdot \mathbf{M}' = -\{[\rho]\}_1 - \{\rho\}_2 + \mathcal{W}' \tag{3.40}$$

$$\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' = -\{[\mathbf{M}]\}_1 - \{\mathbf{M}\}_2 - \{\mathbf{M}\}_3 - \{[\mathbf{M}]\}_2 + \boldsymbol{\mathcal{F}}' \tag{3.41}$$

$$\bar{\rho}C_v\frac{\partial T'}{\partial t} + \bar{p}\nabla \cdot \mathbf{M}' = -\{[T]\}_1 - \{T\}_2 - \{T\}_3 - [\{T\}_2] + \mathcal{Q}' \tag{3.42}$$

$$\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla \cdot \mathbf{M}' = -\{[p]\}_1 - \{p\}_2 + \mathcal{P}' \tag{3.43}$$

$$\bar{\rho}\bar{T}\frac{\partial s'}{\partial t} = -\{[s]\}_1 - \{s\}_2 - \{[s]\}_2 - \{s\}_3 - \{s\}_4 + \mathcal{S}' \tag{3.44}$$

The various brackets are defined in Section A.6 of Annex A. They are formed to contain terms ordered with respect to both the mean Mach number and the amplitude of the fluctuations:

$$\begin{aligned}
\{[\quad]\}_1 &: 1^{st} \text{ order in } \bar{\mathbf{M}};\ 1^{st} \text{ order in } \mathbf{M}',\ O(\mu\varepsilon)\\
\{\quad\}_2 &: 0^{th} \text{ order in } \bar{\mathbf{M}};\ 2^{nd} \text{ order in } \mathbf{M}',\ O(\varepsilon^2)\\
\{[\quad]\}_2 &: 1^{st} \text{ order in } \bar{\mathbf{M}};\ 2^{nd} \text{ order in } \mathbf{M}',\ O(\mu\varepsilon^2)\\
\{\quad\}_3 &: 0^{th} \text{ order in } \bar{\mathbf{M}};\ 3^{rd} \text{ order in } \mathbf{M}',\ O(\varepsilon^3)\\
\{\quad\}_4 &: 0^{th} \text{ order in } \bar{\mathbf{M}};\ 4^{th} \text{ order in } \mathbf{M}',\ O(\varepsilon^4)
\end{aligned} \tag{3.45}$$

No terms have been dropped in passage from the set (3.15)–(3.19) to the set (3.40)–(3.44), but fluctuations of the sources $\mathcal{W}', \cdots, \mathcal{S}'$ are not now classified into the various types defined by the brackets (3.45).

We have put the equations in the forms (3.40)–(3.44) to emphasize the point of view that we are considering classes of problems closely related to motions in classical acoustics. If the right hand sides are ignored, (3.40)–(3.44) become the equations for linear acoustics of a uniform non-reacting medium at rest. The perturbations of that limiting class arise from four types of processes:

(i) interactions of the linear acoustic field with the mean flow, represented by the terms contained in the square brackets within curly brackets, $\{[\quad]\}_1$;

(ii) nonlinear interactions between the fluctuations, represented by the curly brackets conveniently referred to as: $\{\quad\}_2$, second order acoustics; $\{\quad\}_3$, third order acoustics; and $\{\quad\}_4$, fourth order acoustics;

(iii) interactions between the mean flow and nonlinear acoustics to second order, represented by $\{[\quad]\}_2$;

(iv) sources associated with combustion processes, represented by the source terms $\mathcal{W}', \boldsymbol{\mathcal{F}}', \mathcal{Q}', \mathcal{P}'$ and $\mathcal{S}'$.

By selectively retaining one or more of these types of perturbations we define a hierarchy of problems of unsteady motions in combustors. We label these classes of problems O, I, II, III, IV according to the orders

of terms retained in the right hand sides: the left hand sides contain only the terms of order $\varepsilon = \mathbf{M}'_r$, the equations for classical linear acoustics.

O. **Classical Acoustics**, $(\mu = 0, \varepsilon \to 0)$

Perturbations to first order in ε are retained on the right-hand sides of (3.40)–(3.44):

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla \cdot \mathbf{M}' &= \mathcal{W}' \\
\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' &= \boldsymbol{\mathcal{F}}' \\
\bar{\rho}C_v\frac{\partial T'}{\partial t} + \bar{p}\nabla \cdot \mathbf{M}' &= \mathcal{Q}' \\
\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla \cdot \mathbf{M}' &= \mathcal{P}' \\
\bar{\rho}\bar{T}\frac{\partial s'}{\partial t} &= \mathcal{S}'
\end{aligned} \tag{3.46 a-e}$$

I. **Linear Stability**, $O(\varepsilon, \mu\varepsilon)$

Retain interactions linear in the average Mach number and in the fluctuations on the right-hand sides:

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla \cdot \mathbf{M}' &= -\{[\rho]\}_1 + \mathcal{W}' \\
\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' &= -\{[\mathbf{M}]\}_1 + \boldsymbol{\mathcal{F}}' \\
\bar{\rho}C_v\frac{\partial T'}{\partial t} + \bar{p}\nabla \cdot \mathbf{M}' &= -\{[T]\}_1 + \mathcal{Q}' \\
\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla \cdot \mathbf{M}' &= -\{[p]\}_1 + \mathcal{P}' \\
\bar{\rho}\bar{T}\frac{\partial s'}{\partial t} &= -\{[s]\}_1 + \mathcal{S}'
\end{aligned} \tag{3.47 a-e}$$

II. **Second Order Acoustics**, $O(\varepsilon, \mu\varepsilon, \varepsilon^2)$

Retain the linear interactions and the nonlinear second order acoustics on the right-hand sides:

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla \cdot \mathbf{M}' &= -(\{[\rho]\}_1 + \{\rho\}_2) + \mathcal{W}' \\
\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' &= -(\{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2) + \boldsymbol{\mathcal{F}}' \\
\bar{\rho}C_v\frac{\partial T'}{\partial t} + \bar{p}\nabla \cdot \mathbf{M}' &= -(\{[T]\}_1 + \{T\}_2) + \mathcal{Q}' \\
\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla \cdot \mathbf{M}' &= -(\{[p]\}_1 + \{p\}_2) + \mathcal{P}' \\
\bar{\rho}\bar{T}\frac{\partial s'}{\partial t} &= -(\{[s]\}_1 + \{s\}_2) + \mathcal{S}'
\end{aligned} \tag{3.48 a-e}$$

III. **Third Order Acoustics**, $O(\varepsilon, \mu\varepsilon, \varepsilon^2, \varepsilon^3)$

Retain the linear interactions and the nonlinear acoustics up to third order on the right-hand sides:

$$\begin{aligned}
\frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla \cdot \mathbf{M}' &= -(\{[\rho]\}_1 + \{\rho\}_2) + \mathcal{W}' \\
\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' &= -(\{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2 + \{\mathbf{M}\}_3) + \boldsymbol{\mathcal{F}}' \\
\bar{\rho}C_v\frac{\partial T'}{\partial t} + \bar{p}\nabla \cdot \mathbf{M}' &= -(\{[T]\}_1 + \{T\}_2 + \{T\}_3) + \mathcal{Q}' \\
\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla \cdot \mathbf{M}' &= -(\{[p]\}_1 + \{p\}_2) + \mathcal{P}' \\
\bar{\rho}\bar{T}\frac{\partial s'}{\partial t} &= -(\{[s]\}_1 + \{s\}_2 + \{s\}_3) + \mathcal{S}'
\end{aligned} \tag{3.49 a-e}$$

IV. **Mean Flow/Nonlinear Acoustic Interactions**, $O(\varepsilon, \mu\varepsilon, \varepsilon^2, \varepsilon^3, \mu\varepsilon^2)$

Retain all terms on the right-hand sides of (3.40)–(3.44) except $\{s\}_4$ in (3.44). No results have been obtained for this class of problems.

Several other classes of problems possible to define in this context will not be considered here since no results have been reported. There is some indication that problems in class IV may have some important consequences but no theoretical results exist.

In each class of problems, the source terms $\mathcal{W}', \cdots$ must be expanded to orders consistent with the orders of the fluid-mechanical perturbations retained.

3.4. Nonlinear Wave Equations for the Pressure Field

Practically all of the subsequent material in this book will be either directly concerned with pressure waves, or with interpretations of behavior related to pressure waves. The presence of unsteady vorticity causes important revisions of such a restricted point of view, as we have already mentioned, but the basic ideas remain in any event. Hence the wave equation for pressure fluctuations occupies a meaningful position in all five classes of problems defined in the preceding section. Its formation follows the same procedure used in classical acoustics.

Define $\boldsymbol{\mathcal{M}}$ and $\mathcal{R}$ to contain all possible terms arising in the sets of equations constructed for the problems O–IV:

$$\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' = -\boldsymbol{\mathcal{M}} + \boldsymbol{\mathcal{F}}' \tag{3.50}$$

$$\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla \cdot \mathbf{M}' = -\mathcal{R} + \mathcal{P}' \tag{3.51}$$

where

$$\begin{aligned}
\boldsymbol{\mathcal{M}} &= \{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2 + \{\mathbf{M}\}_3 + \{[\mathbf{M}]\}_2 \\
\mathcal{R} &= \{[p]\}_1 + \{p\}_2
\end{aligned} \tag{3.52 a,b}$$

Differentiate (3.51) with respect to time and substitute (3.50) for $\partial \mathbf{M}'/\partial t$:

$$\frac{\partial^2 p'}{\partial t^2} + \gamma \bar{p} \nabla \cdot \left[-\frac{1}{\bar{\rho}} \nabla p' - \frac{1}{\bar{\rho}} \left(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{F}}' \right) \right] = -\frac{\partial \mathcal{R}}{\partial t} + \frac{\partial \mathcal{P}'}{\partial t}$$

Rearrange the equation to find

$$\nabla^2 p' - \frac{1}{\bar{a}^2} \frac{\partial^2 p'}{\partial t^2} = h \tag{3.53}$$

with

$$h = -\bar{\rho} \nabla \cdot \left[\frac{1}{\bar{\rho}} \left(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{F}}' \right) \right] + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} (\mathcal{R} - \mathcal{P}') + \frac{1}{\bar{\rho}} \nabla \bar{\rho} \cdot \nabla p' \tag{3.54}$$

The boundary condition for the pressure field is found by taking the scalar product of the outward normal, at the chamber boundary, with (3.50):

$$\hat{\mathbf{n}} \cdot \nabla p' = -f \tag{3.55}$$

$$f = \bar{\rho} \frac{\partial \mathbf{M}'}{\partial t} \cdot \hat{\mathbf{n}} + \left(\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{F}}' \right) \cdot \hat{\mathbf{n}} \tag{3.56}$$

Replacing $\boldsymbol{\mathcal{M}}$ and $\mathcal{R}$ by their definitions (3.52)a,b, we have the formulation based on the inhomogeneous nonlinear wave equation and its boundary condition:

$$\begin{aligned} \nabla^2 p' - \frac{1}{\bar{a}^2} \frac{\partial^2 p'}{\partial t^2} &= h \\ \hat{\mathbf{n}} \cdot \nabla p' &= -f \end{aligned} \tag{3.57)a,b}$$

with

$$\begin{aligned} h = &\left[-\bar{\rho} \nabla \cdot \frac{1}{\bar{\rho}} \{[\mathbf{M}]\}_1 + \frac{1}{\bar{a}^2} \frac{\partial \{[p]\}_1}{\partial t} + \frac{1}{\bar{\rho}} \nabla \bar{\rho} \cdot \nabla p' \right] + \left[-\bar{\rho} \nabla \cdot \frac{1}{\bar{\rho}} \{\mathbf{M}\}_2 + \frac{1}{\bar{a}^2} \frac{\partial \{p\}_2}{\partial t} \right] \\ &- \left[\bar{\rho} \nabla \cdot \frac{1}{\bar{\rho}} \{\mathbf{M}\}_3 \right] - \left[\bar{\rho} \nabla \cdot \frac{1}{\bar{\rho}} \{[\mathbf{M}]\}_2 \right] + \left[+\bar{\rho} \nabla \cdot \frac{1}{\bar{\rho}} \boldsymbol{\mathcal{F}}' - \frac{1}{\bar{a}^2} \frac{\partial \mathcal{P}'}{\partial t} \right] \end{aligned} \tag{3.58}$$

$$f = \bar{\rho} \frac{\partial \mathbf{M}'}{\partial t} \cdot \hat{\mathbf{n}} + \hat{\mathbf{n}} \cdot \left[\{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2 + \{\mathbf{M}\}_3 + \{[\mathbf{M}]\}_2 \right] - \boldsymbol{\mathcal{F}}' \cdot \hat{\mathbf{n}} \tag{3.59}$$

With this formulation, the wave equations and boundary conditions for the classes of problems defined in Section 3.3 are distinguished by the following functions h and f:

O. **Classical Acoustics**

$$\begin{aligned} h_O &= \bar{\rho} \nabla \cdot \frac{1}{\bar{\rho}} \boldsymbol{\mathcal{F}}' - \frac{1}{\bar{a}^2} \frac{\partial \mathcal{P}'}{\partial t} \\ f_O &= \bar{\rho} \frac{\partial \mathbf{M}'}{\partial \mathbf{t}} \cdot \hat{\mathbf{n}} - \boldsymbol{\mathcal{F}}' \cdot \hat{\mathbf{n}} \end{aligned} \tag{3.60)a,b}$$

I. **Linear Stability**

$$h_I = \left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{[\mathbf{M}]\}_1 + \frac{1}{\bar{a}^2}\frac{\partial\{[p]\}_1}{\partial t} + \frac{1}{\bar{\rho}}\nabla\bar{\rho}\cdot\nabla p'\right] + \left[\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\boldsymbol{\mathcal{F}}' - \frac{1}{\bar{a}^2}\frac{\partial\mathcal{P}'}{\partial t}\right]$$

$$f_I = \bar{\rho}\frac{\partial\mathbf{M}'}{\partial t}\cdot\hat{\mathbf{n}} + \hat{\mathbf{n}}\cdot\{[\mathbf{M}]\}_1 - \boldsymbol{\mathcal{F}}'\cdot\hat{\mathbf{n}} \tag{3.61)a,b}$$

Allowing $\boldsymbol{\mathcal{F}}'$ and $\mathcal{P}'$ to be non-zero gives the opportunity for representing sources of mass, momentum, and energy both within the volume and at the boundary. The first term in f_I accounts for motion of the boundary.

II. **Second Order Acoustics**

$$h_{II} = \left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{[\mathbf{M}]\}_1 + \frac{1}{\bar{a}^2}\frac{\partial\{[p]\}_1}{\partial t} + \frac{1}{\bar{\rho}}\nabla\bar{\rho}\cdot\nabla p'\right] + \left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{\mathbf{M}\}_2 + \frac{1}{\bar{a}^2}\frac{\partial\{p\}_2}{\partial t}\right]$$
$$+ \left[\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\boldsymbol{\mathcal{F}}' - \frac{1}{\bar{a}^2}\frac{\partial\mathcal{P}'}{\partial t}\right] \tag{3.62)a,b}$$

$$f_{II} = \bar{\rho}\frac{\partial\mathbf{M}'}{\partial t}\cdot\hat{\mathbf{n}} + \hat{\mathbf{n}}\cdot[\{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2] - \boldsymbol{\mathcal{F}}'\cdot\hat{\mathbf{n}}$$

III. **Third Order Acoustics**

$$h_{III} = \left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{[\mathbf{M}]\}_1 + \frac{1}{\bar{a}^2}\frac{\partial\{[p]\}_1}{\partial t} + \frac{1}{\bar{\rho}}\nabla\bar{\rho}\cdot\nabla p'\right] + \left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{\mathbf{M}\}_2 + \frac{1}{\bar{a}^2}\frac{\partial\{p\}_2}{\partial t}\right]$$
$$+ \left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{\mathbf{M}\}_3\right] + \left[\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\boldsymbol{\mathcal{F}}' - \frac{1}{\bar{a}^2}\frac{\partial\mathcal{P}'}{\partial t}\right] \tag{3.63)a,b}$$

$$f_{III} = \bar{\rho}\frac{\partial\mathbf{M}'}{\partial t}\cdot\hat{\mathbf{n}} + \hat{\mathbf{n}}\cdot[\{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2 + \{\mathbf{M}\}_3] - \boldsymbol{\mathcal{F}}'\cdot\hat{\mathbf{n}}$$

IV. **Mean Flow/Nonlinear Acoustics Interactions**

$$h_{IV} = h_{III} - \bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{[\mathbf{M}]\}_2$$

$$f_{IV} = f_{III} + \hat{\mathbf{n}}\cdot\frac{\mathbf{1}}{\bar{\rho}}\{[\mathbf{M}]\}_{\mathbf{2}} \tag{3.64)a,b}$$

With these forms for the functions h and f, the definitions of the classes of problems considered here are complete, giving the basis for the analysis worked out in the remainder of this book. Only problems within classical acoustics can be solved easily. All others require approximations, both in modeling physical processes and in the method of solution. Modeling will be discussed in the contexts of specific applications; a few remarks help clarify the approximate method of solution described in the following chapter.

Remarks:

(i) The equations derived here are written with the dimensionless variables defined in Section 3.1. This is an important point to recall when results are written in terms of dimensionless variables. Annex D contains the corresponding results in dimensional variables.

(ii) The classes of problems O–IV defined here are described by inhomogeneous equations that even for linear stability cannot be generally solved in closed form. The chief obstacles to solution arise because the functions h and f contain not only the unknown pressure but also the velocity and temperature. For given functions $\boldsymbol{\mathcal{F}}'$ and $\mathcal{P}'$, numerical solutions could be obtained for a specified combustor and mean flow field. The results would apply only to the special case considered. To obtain some understanding of general behavior it would be necessary to consider many special cases, a tedious and expensive procedure.

(iii) Therefore, we choose to work out an approximate method of solution applicable to all classes of problems. Numerical solutions, or 'simulations' then serve the important purpose of assessing the validity and accuracy of the approximate results.

(iv) The approximate method of solution described in the following chapter is based first on spatial averaging, followed by an iteration procedure involving extension of the expansion in two small parameters defined in this chapter. This method has been most widely used and confirmed in applications to combustion instabilities in solid propellant rockets, but it can be applied to problems arising in any type of combustor.

(v) Instabilities in solid rockets have been particularly helpful in developing the general theory for at least three reasons: 1) the mean flow field, nonuniform and generated by mass addition at the boundary, requires careful attention to processes associated with interactions between the mean flow and unsteady motions; 2) more experimental results for transient behavior have been obtained for solid rockets than for any other combustion system; and 3) although still far from being satisfactorily understood, the dynamics of burning solid propellants is better known than for any other combustion system.

(vi) The fluctuations of the source terms, $\mathcal{W}'$, $\boldsymbol{\mathcal{F}}'$, ... $\mathcal{S}'$ will be made explicit as required in particular applications.

(vii) No assumptions have been made restricting either the average or the time-dependent velocity fields to be irrotational. Moreover, all viscous effects can be accommodated with suitable definitions of the source terms.

(viii) For reasons explained earlier, the wave equations are written for the pressure which, in lowest approximation, is associated only with acoustic waves. However, on the right-hand sides (i.e., in the functions h and f) the total unsteady velocity appears. Hence by suitable decomposition (see Section 7.9) coupling between, say, vorticity and acoustic waves can be investigated. In particular, this formulation allows calculation of the effects of vorticity on stability (Flandro 1995). However, we must emphasize that the methods and results worked out in the following chapters are intended to be relatively easy to apply. The price of this property is their approximate character.

(ix) The most significant omission at this stage is accounting for turbulence. In principle, modeling of turbulence could be included in the derivation of the general equations. However, that strategy would bring unnecessary complications and erect serious obstacles to obtaining useful results with minimal effort. For applications to practical situations, the effects of turbulence seem to be definitely secondary. Theoretical justification for ignoring the possible effects of random or statistical fluctuations in the flow is based on the work of Chu and Kovasznay (1957).