CHAPTER 4

Modal Expansion and Spatial Averaging; An Iterative Method of Solution

From the point of view expressed by Figure 1.1, we are concerned in this chapter with representing the combustor dynamics. The procedure, often called 'modeling', is based on the equations of motion constructed in the preceding section and hence in principle will contain all relevant physical processes.[1] For the purposes here, all modeling of combustor dynamics and of combustion dynamics—the mechanisms and feedback in Figure 1.1—must be done in the context developed in Chapter 1. Thus we always have in mind the idea of wave motions somehow generated and sustained by interactions between the motions themselves and combustion processes, the latter also including certain aspects of the mean flow within the combustor.

The simplest model of the combustor dynamics is a single stationary wave, a classical acoustic resonance as in an organ pipe, but decaying or growing due to the other processes in the chamber. In practice, the combustion processes and nonlinear gasdynamical effects inevitably lead to the presence of more than one acoustic mode. We need a relatively simple yet accurate means of treating those phenomena for problems of the sort arising in the laboratory and in practice. Modeling in this case begins with construction of a suitable method for solving the nonlinear wave equations derived in Section 3.4. In this context we may regard the analysis of the Rijke tube covered in Section 2.7 as a basic example of the procedure stripped of the formalism covered in this chapter.

The chief purpose of the analysis constructed here is to devise methods capable of producing results useful for prediction and interpretation of unsteady motions in full-scale combustion chambers, as well as for laboratory devices. That intention places serious demands on the methods used for at least two reasons:

(i) processes that must be modeled are usually complicated and their theoretical representations are necessarily approximate to extents which themselves are difficult to assess; and

(ii) almost all input data required for quantitative evaluation of theoretical results are characterized by large uncertainties.

In this situation it seems that for applications and, as it will turn out, for theoretical purposes as well, the most useful methods will be based on some sort of spatial averaging. Formal solution of the partial differential equations, even for linear problems, is practically a hopeless task except for very special cases involving simple geometries. Direct numerical simulations (DNS), or numerical solutions to the partial differential equations, are not real alternatives for practical purposes at this time, and alone are not attractive for obtaining basic understanding. However, as we will see later, numerical methods offer the only means for assessing the validity of approximate solutions, and can always be applied to more complicated (realistic?) problems than we can reasonably handle with the analytical methods discussed here. In any event, one should view theory and analysis on the one hand, and numerical simulations on the other, as complementary activities. Recent experiences have shown that careful coordination of the analytical procedures and numerical simulations with experimental observations is the most effective strategy for treating combustion instabilities in actual

[1] That seems to be what some people (apparently electrical engineers) mean by the term 'physics-based modeling.' What would otherwise be the basis for acceptable modeling of a physical system has not been explained.

combustion systems. With the foundation of basic theory, numerical simulations offer very powerful means for improving understanding.

The greater part of the material on analysis and theory of combustion instabilities in this book is based on a method of spatial averaging. It is important to notice that the elementary example worked out in Section 2.7 already shows the superior results possible with the method of averaging in contrast to an approximate solution not involving averaging. The essential idea is of course not new, the method being nearly identical with similar methods used in other branches of continuum mechanics. There are a few special characteristics associated with applications to combustor that will appear in the course of the following discussion.

4.1. Application of a Green's Function for Steady Waves

The method used later to analyze nonlinear behavior has its origins in an early analysis of linear combustion instabilities in liquid rocket engines (Culick, 1961, 1963). That work was based on solution to problems of steady waves by introducing a Green's function. It is an effective strategy for this application because departures from a known soluble problem are small, due either to perturbations within the volume or at the boundary, all of order μ in the context developed in Chapter 3. Mitchell (1993) has made the most extensive use of Green's functions in this context.

The problem to be solved is defined by the equation derived in Section 3.4,

$$\begin{aligned} \nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} &= h \\ \hat{\mathbf{n}} \cdot \nabla p' &= -f \end{aligned} \tag{4.1)a,b}$$

with $\bar{a}$ constant, and h and f given by (3.61)a,b for linear stability. Because here h and f are assumed linear,[2] various methods are available to build general solutions by applying the principle of superposition to elementary solutions representing steady waves. Hence we assume that the fluctuating pressure field is a steady wave system within the given chamber, having unknown spatial structure varying harmonically in time:

$$p' = \hat{p} e^{-i\bar{a}kt} \tag{4.2}$$

where k is the complex wavenumber, also initially unknown,

$$k = \frac{1}{\bar{a}}(\omega + i\alpha) \tag{4.3}$$

As defined here, α positive means that the wave has growing amplitude, $p' \sim e^{\alpha t}$. Of course the wave is not strictly stationary, a condition existing only if $\alpha = 0$, certainly true when $h = f = 0$, as in classical acoustics.

Even when h, f are non-zero, it is still possible that $\alpha = 0$, now defining a state of *neutral stability.* In general one must expect $\alpha \neq 0$; it is a basic assumption in all of the analysis covered in this book that α is small compared with ω, so the waves are slowly growing or decaying—they are 'almost' stationary, and their spatial structure does not change drastically with time. However, the results obtained are quite robust and seem often to be usable even when α/ω is not small.

In the first instance, the problem here is to determine the spatial distribution $\hat{p}$ and the complex wavenumber k. For steady waves we can write

$$h = \kappa \hat{h} e^{-i\bar{a}kt} \ ; \ f = \kappa \hat{f} e^{-i\bar{a}kt}$$

[2]Sections 4.1 and 4.2 cover linear behavior only; Sections 4.3–4.6 include nonlinear behavior.

where again κ is a small parameter[3] characterizing the smallness of h and f. Substitution in (4.1)a,b and dropping the common exponential time factor gives

$$\begin{aligned} \nabla^2\hat{p} + k^2\hat{p} &= \kappa\hat{h} \\ \hat{\mathbf{n}}\cdot\nabla\hat{p} &= -\kappa\hat{f} \end{aligned} \tag{4.4)a,b}$$

This is of course a well-known classical problem thoroughly discussed in many books. Many methods of solution are available for the linear problem. We use here a procedure based on introducing a Green's function discussed, for example, by Morse and Feshbach (1952, Chapter 9). This is an attractive method for several reasons, including:

(i) Conversion of a differential equation into an integral equation, and the iterative method of solution this suggests, is an effective means for minimizing the consequences of the uncertainties inherent in problems of combustor dynamics;

(ii) Explicit results can be obtained for real and imaginary parts of the complex wavenumber in forms that are easily interpreted and remarkably convenient both for theoretical work and for applications;

(iii) The method has motivated a straightforward extension to nonlinear problems, with considerable success. (Chapter 7)

Define a Green's function satisfying the homogeneous boundary condition and the wave equation homogeneous except at the single point where a source is located, having zero spatial extent and infinite strength such that its integral over space is finite. Thus the source is represented by a delta function $\delta(\mathbf{r}-\mathbf{r}_0)$ and $G(\mathbf{r}|\mathbf{r}_0)$ is determined as a solution to the problem

$$\begin{aligned} \nabla^2 G(\mathbf{r}|\mathbf{r}_0) + k^2 G(\mathbf{r}|\mathbf{r}_0) &= \delta(\mathbf{r}-\mathbf{r}_0) \\ \hat{\mathbf{n}}\cdot\nabla G(\mathbf{r}|\mathbf{r}_0) &= 0 \end{aligned} \tag{4.5)a,b}$$

The notation $\mathbf{r}|\mathbf{r}_0$ as the argument of $G(\mathbf{r}|\mathbf{r}_0)$ represents the interpretation of the Green's function as the wave observed at point $\mathbf{r}$ due to a steady oscillatory point source at $\mathbf{r}_0$.

Multiply (4.4)a by $G(\mathbf{r}|\mathbf{r}_0)$, (4.5)a by $\hat{p}(\mathbf{r})$, subtract the results and integrate over volume (in the present case the volume of the chamber) to find

$$\begin{aligned} \iiint_V \left[G(\mathbf{r}|\mathbf{r}_0)\nabla^2\hat{p}(\mathbf{r}) - \hat{p}(\mathbf{r})\nabla^2 G\,(\mathbf{r}|\mathbf{r}_0)\right] dV + k^2 \iiint_V \left[G(\mathbf{r}|\mathbf{r}_0)\hat{p}(\mathbf{r}) - \hat{p}(\mathbf{r})G(\mathbf{r}|\mathbf{r}_0)\right] dV \\ = \kappa \iiint_V G(\mathbf{r}|\mathbf{r}_0)\hat{h}(\mathbf{r})dV - \iiint_V \hat{p}(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)dV \end{aligned} \tag{4.6}$$

Because $G(\mathbf{r}|\mathbf{r}_0)$ and $\hat{p}(\mathbf{r})$ are scalar functions the second integral on the left-hand side vanishes. The first integral is rewritten using a form of Green's theorem, and the basic property of the delta function is applied to the second integral on the right-hand side:

$$\iiint_V F(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)dV = F(\mathbf{r}_0) \qquad (\mathbf{r}, \mathbf{r}_0 \text{ in } V) \tag{4.7}$$

Hence (4.6) becomes

$$\oiint_S \left[G(\mathbf{r}|\mathbf{r}_0)\nabla\hat{p}(\mathbf{r}) - p(\mathbf{r})\nabla G(\mathbf{r}|\mathbf{r}_0)\right]\cdot\hat{\mathbf{n}}dS = \kappa \iiint_V G(\mathbf{r}|\mathbf{r}_0)\hat{h}(\mathbf{r})dV - \hat{p}(\mathbf{r}_0)$$

where $\hat{\mathbf{n}}$ is the outward normal at the surface of the volume V in question.

[3]Later, κ will be identified with μ introduced in Section 3.3 but it is useful in this discussion to maintain a distinction.

Now apply the boundary conditions (4.4)b and (4.5)b and the last equation can be written in the form

$$\mathbf{p}(\hat{r}_0) = \kappa \left\{ \iiint_V G(\mathbf{r}|\mathbf{r}_0)\hat{h}(\mathbf{r})dV + \oiint_S G(\mathbf{r_s}|\mathbf{r}_0)\hat{f}(\mathbf{r}_s)dS \right\} \tag{4.8}$$

Subscript $(\ )_s$ means the point $\mathbf{r}_s$ lies on the boundary surface (actually on the inside surface of the boundary). Because the operator for scalar waves is *self-adjoint* (see Morse and Feshbach 1953, Chapter 10), the Green's function possesses the property of symmetry

$$G(\mathbf{r}|\mathbf{r}_0) = G(\mathbf{r}_0|\mathbf{r}) \tag{4.9}$$

This property has the appealing physical interpretation that the wave observed at $\mathbf{r}$ due to a point source at $\mathbf{r}_0$ has the same amplitude and relative phase as for the wave observed at $\mathbf{r}_0$ when the point source is located at $\mathbf{r}$. With (4.9) we can interchange $\mathbf{r}$ and $\mathbf{r}_0$ in (4.8) to find for the steady field at position $\mathbf{r}$:

$$\hat{p}(\mathbf{r}) = \kappa \left\{ \iiint_V G(\mathbf{r}|\mathbf{r}_0)\hat{h}(\mathbf{r}_0)d\mathbf{V}_0 + \oiint_S G(\mathbf{r}|\mathbf{r}_{0s})\hat{f}(\mathbf{r}_{0s})dS_0 \right\} \tag{4.10}$$

Equation (4.10) is not an explicit solution for the pressure field, but is rather an integral equation, because the source functions $\hat{h}$ and $\hat{f}$ in general depend on the fluctuating pressure and velocity fields themselves. However, because the sources are assumed to be small perturbations of the classical field having no sources, κ is small and $\hat{p}$ will not differ greatly from a solution to the homogeneous problem defined by $h = f = 0$. The result (4.10) represents the solution to the inhomogeneous problem; the complete solution is (4.10) plus a homogeneous solution. We will take advantage of the smallness of κ to find an approximate explicit solution for $\hat{p}$ by an iterative procedure discussed in Section 4.1.1.

Whatever tactic one may choose to follow, the result (4.10) is of no practical value without having a representation of $G(\mathbf{r}|\mathbf{r}_0)$. The most convenient form of $G(\mathbf{r}|\mathbf{r}_0)$ for our purpose is expansion in eigenfunctions $\psi_n(\mathbf{r})$, here the normal modes of the classical acoustics problem with no sources in the volume and homogeneous boundary conditions; $G(\mathbf{r}|\mathbf{r}_0)$ is therefore expressed as a *modal expansion,*

$$G(\mathbf{r}|\mathbf{r}_0) = \sum_{n=0}^{\infty} A_n \psi_n(\mathbf{r}) \tag{4.11}$$

where the ψ_n satisfy[4]

$$\begin{aligned} \nabla^2\psi_n + k_n^2\psi_n &= 0 \\ \hat{\mathbf{n}} \cdot \nabla\psi_n &= 0 \end{aligned} \tag{4.12)a,b}$$

and are orthogonal functions,

$$\iiint_V \psi_m(\mathbf{r})\psi_n(\mathbf{r})dV = E_n^2\delta_{mn} \tag{4.13}$$

Substitute (4.11) in (4.5)a, multiply by $\psi_m(\mathbf{r})$ and integrate over the volume to find

$$\iiint_V \psi_m \sum_{n=0}^{\infty} A_n \nabla^2\psi_n dV + k^2 \iiint_V \psi_m \sum_{n=0}^{\infty} A_n\psi_n dV = \iiint_V \psi_m(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)dV$$

With (4.7), (4.12) and (4.13), this equation produces the formula for A_n:

$$A_n = \frac{\psi_n(\mathbf{r}_0)}{k_n^2 - k^2} \tag{4.14}$$

[4]Equations (4.12)a,b really are essential to the following general results. They can be altered, e.g. $\hat{n} \cdot \nabla\psi_n = -g(\mathbf{r})$, but subsequent formulas must be carefully checked.

Thus the expansion (4.11) for $G(\mathbf{r}|\mathbf{r}_0)$ is

$$G(\mathbf{r}|\mathbf{r}_0) = \sum_{n=0}^{\infty} \frac{\psi_n(\mathbf{r})\psi_n(\mathbf{r}_0)}{E_n^2(k^2 - k_n^2)} \tag{4.15}$$

the modal expansion of the Green's function. Substitution of (4.15) in (4.10) leads to the formal modal expansion of the pressure field,[5]

$$\hat{p}(\mathbf{r}) = \kappa \sum_{n=0}^{\infty} \frac{\psi_n(\mathbf{r})}{E_n^2(k^2 - k_n^2)} \left\{ \iiint_V \psi_n(\mathbf{r}_0)\hat{h}(\mathbf{r}_0)dV_0 + \oiint_S \psi_n(\mathbf{r}_{0s})\hat{f}(\mathbf{r}_{0s})dS_0 \right\} \tag{4.16}$$

Suppose that for κ tending to zero, $\hat{p}(\mathbf{r})$ approaches the unperturbed mode shape ψ_N; let the corresponding function $\hat{p}$ be denoted $\hat{p}_N$, so

$$\hat{p} \underset{\kappa \to 0}{\longrightarrow} \hat{p}_N = \psi_N \tag{4.17}$$

Now separate the Nth term from the sum in (4.16) and write

$$\begin{aligned} \hat{p}(\mathbf{r}) = \psi_N(\mathbf{r}) \frac{\kappa}{E_N^2(k^2 - k_N^2)} & \left\{ \iiint_V \psi_N(\mathbf{r}_0)\hat{h}(\mathbf{r}_0)dV_0 + \oiint_S \psi_N(\mathbf{r}_{0s})\hat{f}(\mathbf{r}_{0s})dS_0 \right\} \\ + \kappa \sum_{n=0}^{\infty}{}' \frac{\psi_n(\mathbf{r})}{E_n^2(k^2 - k_n^2)} & \left\{ \iiint_V \psi_n(\mathbf{r}_0)\hat{h}(\mathbf{r}_0)dV_0 + \oiint_S \psi_n(\mathbf{r}_{0s})\hat{f}(\mathbf{r}_{0s})dS_0 \right\} \end{aligned} \tag{4.18}$$

where the prime in the summation sign means that the term $n = N$ is missing. The eigenvalues associated with the eigenfunction $\psi_N(\mathbf{r})$ is k_N. This form is consistent with the requirement (4.17) only if the factor multiplying $\psi_N(\mathbf{r})$ is unity, giving the formula for the perturbed wavenumber

$$k^2 = k_N^2 + \frac{\kappa}{E_N^2} \left\{ \iiint_V \psi_N(\mathbf{r}_0)\hat{h}(\mathbf{r}_0)dV_0 + \oiint_S \psi_N(\mathbf{r}_{0s})\hat{f}(\mathbf{r}_{0s})dS_0 \right\} \tag{4.19}$$

and (4.18) becomes

$$\hat{p}(\mathbf{r}) = \psi_N(\mathbf{r}) + \kappa \sum_{n=0}^{\infty}{}' \frac{\psi_n(\mathbf{r})}{E_n^2(k^2 - k_n^2)} \left\{ \iiint_V \psi_n(\mathbf{r}_0)\hat{h}(\mathbf{r}_0)dV_0 + \oiint_S \psi_n(\mathbf{r}_{0s})\hat{f}(\mathbf{r}_{0s})dS_0 \right\} \tag{4.20}$$

Another more direct derivation of (4.19) very useful in later analysis, may be had by first multiplying (4.4) by ψ_N and integrating over the volume:

$$\iiint_V \psi_N \nabla^2 \hat{p} dV + k^2 \iiint_V \psi_N \hat{p} dV = \kappa \iiint_V \psi_N \hat{h} dV$$

Application of Green's theorem to the first integral gives

$$\iiint_V \hat{p}\nabla^2\psi_N dV + \oiint_S [\psi_n \nabla\hat{p} - \hat{p}\nabla\psi_n] \cdot \hat{\mathbf{n}} dS + k^2 \iiint_V \psi_n \hat{p} dV = \iiint_V \psi_N \hat{h} dV$$

[5]The form of (4.16) has been seriously misunderstood by many interested in methods for analyzing and interpreting combustion instabilities. According to (4.12)b, the velocity associated with each of the basis functions must vanish on the surface enclosing the volume considered. Hence the representation (4.16) seems also to imply that the velocity of the actual (perturbed) field must also vanish at the boundary, even with $\hat{h}$ and $\hat{f}$ non-zero. That is, the approximate solution for $p(\mathbf{r}, t)$ and $u(\mathbf{r}, t)$ cannot satisfy the correct (perturbed) boundary conditions. This conclusion is incorrect, following from the implied assumption that as $\mathbf{r} \to \mathbf{r}_0$ on the boundary, the limit as $\mathbf{r} \to \mathbf{r}_0$ in the sum (4.16) is equal to the sum of the limits of each of the terms in (4.16). The point is made by example in Annex F; see also Footnote 8 and related remarks at the end of this chapter.

after inserting $\nabla^2\psi_N = -k_N^2\psi_N$ and $\nabla\psi_N \cdot \hat{\mathbf{n}} = 0$, rearrangement gives

$$k^2 = k_N^2 + \frac{\kappa}{\iiint\limits_V \psi_N \hat{p} dV} \left\{ \iiint\limits_V \psi_N(\mathbf{r})\hat{h}(\mathbf{r})dV + \oiint\limits_S \psi_N(\mathbf{r}_s)\hat{f}(\mathbf{r}_s)dS \right\} \tag{4.21}$$

The integral of $\psi_N\hat{p}$ in the denominator of (4.21) can be evaluated by using (4.20) and is exactly E_N^2, providing the series in (4.20) converges. Hence (4.21) is identical to (4.19). This simple calculation has shown that (4.18) and (4.20) are consistent.

The preceding calculation contains several basic ideas lying behind much of the analysis used in this book. In summary, the original problem described by the differential equation (4.4)a and its boundary condition (4.4) are converted to an integral equation, in this case (4.10), established by introducing a Green's function. This is not an explicit solution because the functions $\hat{h}$ and $\hat{f}$ generally depend on the dependent variable $\hat{p}$. However, formulation as an integral equation provides a convenient basis for approximate solution by iteration.

4.1.1. Approximate Solution by Iteration. To apply an iterative procedure, it is necessary first to give the Green's function $G(\mathbf{r}|\mathbf{r}_0)$ explicit form. The natural choice for problems of waves in a chamber is a series expansion in the natural modes of the chamber, a modal expansion, (4.15). For the small parameter κ tending to zero (i.e. all perturbations of the classical acoustics problem are small), a straightforward argument produces the formula (4.19) for the wavenumber and the integral equation (4.20) for $\hat{p}(\mathbf{r})$.

Apparently, equation (4.20) must be solved to give $\hat{p}$ before the wavenumber can be computed with (4.19). We should emphasize that for many practical purposes, it is really k that is required, because its imaginary part determines the linear stability of the system ($\alpha = 0$). The great advantage of this approach may be seen clearly with a simple example. Suppose $\hat{f} = 0$ and $\hat{h} = H(\mathbf{r})\hat{p}$ in (4.4)a,b. Then (4.20) and (4.19) become

$$\hat{p}(\mathbf{r}) = \psi_N(\mathbf{r}) + \kappa \sum_{n=0}^{\infty}{}' \frac{\psi_n(\mathbf{r})}{E_n^2(k^2 - k_n^2)} \iiint\limits_V \psi_n H(\mathbf{r}_0)\hat{p}(\mathbf{r}_0)dV_0 \tag{4.22}$$

$$k^2 = k_N^2 + \frac{\kappa}{E_N^2} \iiint\limits_V \psi_N H(\mathbf{r}_0)\hat{p}(\mathbf{r}_0)dV_0 \tag{4.23}$$

Because κ is assumed to be small, solution by successive approximation, i.e. an iterative procedure, is a natural way to proceed. The initial (zeroth) approximation to the mode shape $\hat{p}$ is (4.22) with $\kappa = 0$, $\hat{p}^{(0)} = \psi_N$. Substitution in (4.23) gives k^2 correct to first order in κ:

$$\begin{aligned}(k^2)^{(1)} &= k_N^2 + \frac{\kappa}{E_N^2} \iiint\limits_V H(\mathbf{r}_0)\psi_N^2 dV_0 \\ &= k_N^2 + \kappa\frac{I_N}{E_N^2}\end{aligned} \tag{4.24}$$

where I_N stands for the integral,

$$I_N = \iiint\limits_V H(\mathbf{r}_0)\psi_N^2 dV_0$$

Calculation of $\hat{p}$ to *first* order in κ requires setting $\hat{p}$ and k^2 to their *zeroth* order values on the right-hand side of (4.22), $\hat{p}^{(0)} = \psi_N$, $(k^2)^{(0)} = k_N^2$:

$$\begin{aligned}\hat{p}^{(1)}(\mathbf{r}) &= \psi_N(\mathbf{r}) + \kappa \sum_{n=0}^{\infty}{}' \frac{\psi_n(\mathbf{r})}{E_n^2(k_N^2 - k_n^2)} \iiint_V \psi_n H(\mathbf{r}_0)\psi_N dV_0 \\ &= \psi_N + \kappa\sigma_N\end{aligned}$$

and σ_N stands for the series

$$\sigma_N = \sum_{n=0}^{\infty}{}' \frac{\psi_N}{E_n^2(k_N^2 - k_n^2)} \iiint_V H(\mathbf{r}_0)\psi_N^2(\mathbf{r}_0)dV_0$$

Substitution of this formula for $\hat{p}$ under the integral in (4.23) then gives the second approximation $(k^2)^{(2)}$ to k^2:

$$\begin{aligned}(k^2)^{(2)} &= k_N^2 + \frac{\kappa}{E_N^2} \iiint_V \psi_N H(\mathbf{r}_0)(\psi_N + \kappa\sigma_N)dV_0 \\ &= (k^2)^{(1)} + \kappa^2 \frac{\kappa}{E_N^2} \iiint_V \psi_N H(\mathbf{r}_0)\sigma_N dV_0\end{aligned} \tag{4.25}$$

A wonderful property of the procedure is already apparent: **Calculation of the wavenumber to some order l in the small parameter requires knowing the modal functions only to order $l-1$**. That is the basis for the current standard practice of computing linear stability for solid propellant rockets (the "Standard Stability Prediction Program," Nickerson *et al.* 1984) using the unperturbed acoustic modes computed for the geometry in question.[6,7]

The "perturbation-iteration" procedure just described is an old and widely used method to obtain solutions to nonlinear as well as linear problems. Often much attention is paid to achieving more accurate solutions by carrying the iterations to higher order in the small parameter. That is a legitimate process providing the equations themselves are valid to the order sought. In Chapter 3 we emphasized the importance of the expansion procedure largely for that reason. If the equations are valid, say, only to second order in the amplitude (ε), there is no need—in fact no justification—to try to find a solution to order ε^3 and higher. Similar remarks apply to the expansion in the average Mach number (μ); see footnote 2 in Chapter 3. The procedure is fully explained in Section 4.6 for the equations derived in Section 3.4.

4.2. An Alternative Derivation of the First Order Formula

The results (4.19) and (4.20) for the complex wavenumber and mode shape can be instructively obtained in a different way. Both formulas provide means for computing the differences $k^2 - k_N^2$ and $\hat{p} - \psi_N$ between the actual (perturbed) quantities and the unperturbed quantities. It is reasonable that those results should somehow follow from comparison of the perturbed ($\kappa \neq 0$) and unperturbed ($\kappa = 0$) problems. The idea is to average the difference between the two problems weighted respectively by the other's mode shape. That is, subtract $\hat{p}$ times equation (4.12)a from ψ_n times (4.4)a and integrate the result over the volume of the

[6]Failure to respect this basic property of the procedure has rendered useless some discussions of the subject. For example, the lengthy discussion by Van Moorhem (1982) is largely irrelevant to the spacially averaged representation treated here and in earlier works. For reasons different from those offered by Van Moorhem, some of the results in question are incomplete, as discussed in Chapters 6 and 7.

[7]Calculation of the solution, the mode shape to order l, also requires modal functions to order $l-1$; see Section 4.6.2.

chamber:

$$\iiint\limits_V \left[\psi_N \nabla^2 \hat{p} - \hat{p}\nabla^2\psi_N\right] dV + \iiint\limits_V (k^2 - k_N^2)\psi_n \hat{p} dV_0 = \kappa \iiint\limits_V \psi_N \hat{h} dV$$

Now apply Green's theorem to the first integral, substitute the boundary conditions (4.4)b and (4.12)b and rearrange the result to find (4.21):

$$k^2 = k_N^2 + \frac{\kappa}{\iiint\limits_V \psi_N \hat{p} dV} \left\{ \iiint\limits_V \psi_N(\mathbf{r})\hat{h}(\mathbf{r})dV + \oiint\limits_S \psi_N(\mathbf{r}_s)\hat{f}(\mathbf{r}_s)dS \right\} \tag{4.26}$$

If k^2 is to be calculated to first order in κ, then $\hat{p}$ must be replaced by its zero order approximation $\hat{p} = \psi_N$. Because the correction to k_N^2 contains the multiplier κ, any contributions of order κ multiplying κ give terms of order κ^2. Hence to first order, (4.26) of course becomes (4.19).

This approach does not provide a recipe for computing the modal or basis functions to higher order. That does not cause difficulty here because we have the procedure given in the preceding section. We will find later that the simple derivation just given suggests a useful extension to time-dependent nonlinear problems. In that situation there is no result corresponding to (4.20) for computing the mode shapes to higher order. That deficiency is a serious obstacle to further progress with a simplified form of the general procedure, a subject of current research.

4.3. Approximate Solution for Unsteady Nonlinear Motions

The method covered in the preceding two sections, based essentially on the use of Green's functions, was the first application of modal expansions and spatial averaging to combustion instabilities (Culick 1961, 1963). In the early 1970s the procedure was extended to treat nonlinear problems, necessarily involving time-dependence (Culick 1971, 1975). We summarize that approach here; an alternative formulation based on a form of Galerkin's method is discussed in the following section.

We begin with the general problem (4.1)a,b and assume an approximation $\tilde{p}'(\mathbf{r})$ to the pressure field as a truncated expansion in a set of basis functions ψ_m,

$$\tilde{p}'(\mathbf{r}, t) = \bar{p}_r \sum_{m=1}^{M} \eta_m(t)\psi_m(\mathbf{r}) \tag{4.27}$$

where for simplicity—not an essential assumption—we take $\bar{p}_r$ to be the average pressure in the chamber, uniform in space and constant in time. In this work we will always take the ψ_m to be acoustic modes defined by the geometry, the distribution of average temperature and suitable boundary conditions.[8] We would like the right-hand side of (4.27) to become more nearly equal to the actual pressure field in the combustor as more terms are included in the series, so that $\tilde{p}' \equiv p'$ in the limit:

$$p'(\mathbf{r}, t) = \lim_{M\to\infty} \tilde{p}'(\mathbf{r}; t) = \lim_{M\to\infty} \bar{p}_r \sum_{m=1}^{M} \eta_m(t)\psi_m(\mathbf{r}) \tag{4.28}$$

[8]The selection of boundary conditions is part of the art of applying this method. Examples covered later will clarify the point. For the present, it is helpful to think of the ψ_m as classical acoustic modes for a volume having rigid walls and the same shape as the combustion chamber in question. The ψ_m therefore do not satisfy exactly the boundary conditions actually existing in a combustor. Hence the right-hand side of (4.27) is an approximation in three respects: the series is truncated to a finite number of terms; it does not satisfy the correct boundary conditions; and the basis functions are assumed to be solutions to the scalar Helmholtz equation with the homogeneous boundary condition $\hat{\mathbf{n}} \cdot \nabla\psi_n = 0$. The ψ_m alone do not represent solutions with the perturbations taken into account. However, the solution carried out to the next order *does* satisfy the boundary conditions to first order. This important point is discussed in Chapter 10 of Morse and Feshbach (1952). The approximate nature of the modal expansion will be clarified as the analysis proceeds. See also Annex F for an example making the point.

Because the ψ_m do not satisfy the correct boundary conditions, this pointwise property certainly cannot be satisfied at the boundary. It is reasonable, however, to expect convergence in an integral-squared sense; that is, the integral of the square of the difference between the exact solution and (4.27) satisfies

$$\lim_{M\to\infty} \iiint_V \left[p'(\mathbf{r},t) - \bar{p}_r \sum_{m=1}^{M} \eta_m(t)\psi_m(\mathbf{r}) \right]^2 dV = 0 \tag{4.29}$$

We will not prove this properly, but assume its truth.

Convergence in the sense asserted by (4.29) is a common idea arising, for example, in formal treatments of Sturm-Liouville problems; see Hildebrand (1952) for a very readable discussion. The matter of convergence of approximate solutions in the present context is more complicated because one must take into account the fact that the governing equations and their solutions are expanded in the two small parameters μ and ε introduced in Chapter 3. We will also not discuss that problem.

The synthesis of the pressure field expressed by (4.27) does not restrict in any practical fashion the generality of the method. For definitions here we assume that the modal functions satisfy the homogeneous Neumann condition $\hat{\mathbf{n}} \cdot \nabla\psi_n = 0$, but for some applications a different boundary condition, perhaps over only part of the boundary, may serve better. Hence we will assume here that the ψ_n are eigensolutions to the problem (4.12)a,b.

We require that the approximation (4.27) to p' satisfy equation (4.1)a. Multiply (4.12)b written for ψ_N by $\tilde{p}'(\mathbf{r},t)$, subtract from (4.1)a written for $\tilde{p}'$ multiplied by ψ_N; then integrate the difference over the volume of the chamber to give

$$\iiint_V \left[\psi_N \nabla^2 \tilde{p}' - \tilde{p}' \nabla^2 \psi_N\right] dV - \iiint_V \frac{1}{\bar{a}^2}\frac{\partial^2 \tilde{p}'}{\partial t^2} dV - k_N^2 \iiint_V \tilde{p}'\psi_N dV = \iiint_V \psi_N h dV$$

Apply Green's theorem to the first integral, substitute the boundary conditions and rearrange the result to give

$$\iiint_V \frac{1}{\bar{a}^2}\frac{\partial^2 \tilde{p}'}{\partial t^2}\psi_N dV + k_N^2 \iiint_V \tilde{p}'\psi_N dV = -\left\{ \iiint_V h\psi_N dV + \oiint_S f\psi_N dS \right\} \tag{4.30}$$

Now substitute the modal expansion (4.27) in the left-hand side:

$$\frac{\bar{p}_r}{\bar{a}_r^2} \sum_{m=0}^{M} \ddot{\eta}_m(t) \iiint_V \left(\frac{\bar{a}_r}{\bar{a}}\right)^2 \psi_m\psi_N dV - k_n^2 \bar{p} \sum_{m=0}^{M} \eta_m \iiint_V \psi_m\psi_N dV = E_N^2 \frac{\bar{p}}{\bar{a}_r^2} F_N \tag{4.31}$$

where

$$F_N = -\frac{\bar{a}^2}{\bar{p}E_N^2} \left\{ \iiint_V h\psi_N dV + \oiint_S f\psi_N dS \right\} \tag{4.32}$$

and $\bar{a}_r$ is a constant reference speed of sound. The second sum in (4.31) reduces, due to the orthogonality of the ψ_m, to $\eta_n E_n^2$. Under the first integrals, write

$$\Delta_a = 1 - \left(\frac{\bar{a}_r}{\bar{a}}\right)^2 \tag{4.33}$$

Then the first sum in (4.31) is

$$\sum_{m=0}^{M} \ddot{\eta}_m(t) \iiint_V (1-\Delta_a)\psi_m\psi_N dV = E_N^2 \ddot{\eta}_N - \sum_{m=0}^{M} \ddot{\eta}_m(t) \iiint_V \Delta_a \psi_m\psi_N dV \tag{4.34}$$

With these changes, equation (4.31) becomes

$$\ddot{\eta}_N + \omega_N^2 \eta_N = F_N + \frac{1}{E_N^2} \sum_{m=0}^{M} \ddot{\eta}_m(t) \iiint_V \Delta_a \psi_m \psi_N dV \tag{4.35}$$

The sum on the right-hand side represents part of the effect of a non-uniform speed of sound in the chamber (if $\Delta_a \neq 0$). To simplify writing we will ignore this term until we consider special problems. For solid rockets it is normally a negligible contribution. If the combustor contains flame sheets, the temperature is piecewise uniform and this term also doesn't appear, but the presence of the discontinuities generates corresponding terms arising from F_N (see Annex E). Thus there are useful situations in which we deal with the system of equations:

$$\ddot{\eta}_N + \omega_N^2 \eta_N = F_N \tag{4.36}$$

This result, a set of coupled nonlinear equations with the forcing function F_N given by (4.32), is the basis for practically all of the analysis and theory discussed in the remainder of this book. A corresponding result is given in Annex B for a purely one-dimensional formulation. In anticipation of later discussions, several general remarks are in order.

(i) The formulation expressed by (4.36) accommodates all relevant physical processes. In the derivation of the conservation equations in Annex A, only inconsequential approximations (for present purposes) were made, notably the neglect of multi-component diffusion and the representation of the reacting multi-phase medium by a single-fluid model. However, only the basic gasdynamics are known explicitly. All other processes must be modeled in suitable forms.

(ii) Despite the apparent generality of (4.36), attention must be paid to an assumption implied in the application of Green's theorem in spatial averaging. That is, the functions involved must possess certain properties of continuity within the volume of averaging. The condition is not satisfied, for example, at a flame sheet, where the velocity is discontinuous, an important exception. Annex E introduces the method for handling such cases.

(iii) The selection of functions for the modal expansion (4.27) is not unique; possible alternatives must always be considered. What works best depends on the nature of the boundary conditions. The closer the boundary is to a rigid reflecting surface, the more effective is the choice $\hat{\mathbf{n}} \cdot \nabla \psi_n = 0$, meaning that the acoustic velocity vanishes on the boundary. Because a combustor must provide for inflow of reactants and exhaust of products, it is simply not possible that the actual enclosure be everywhere rigid and perfectly reflecting. For $\hat{\mathbf{n}} \cdot \nabla \psi_n = 0$ to be a good approximation, as it should be for the modal expansion to serve successfully as a zeroth approximation to the pressure field, the boundary must be 'nearly' reflecting. Choked inlets and outlets satisfy the condition if the Mach number at the chamber side is small (that is, the flow within the volume is consistent with the assumption $\mu \ll 1$). Also, the dynamical response of burning solid propellants is normally such that requiring $\hat{\mathbf{n}} \cdot \nabla \psi_n = 0$ is appropriate. Over a broad useful range of practical conditions, defining the basis functions with (4.12)a,b is therefore a reasonable choice. Exceptions are not rare, however, and care must be exercised. For example, a Rijke tube will contain a heater, or a thin combustion region within the duct. Continuous functions ψ_n may not be good zeroth approximations to the actual behavior discontinuous at the heating zone; moreover, in that case $\psi_n = 0$ at the ends is the proper choice for boundary conditions on the basis functions. More generally, if the temperature field is highly non-uniform, then the zeroth order expansion functions should take that feature into account.

(iv) An enormous advantage of the result (4.36) is its clear interpretation. A general unsteady motion in a combustor is represented by the time-evolution of a system of coupled nonlinear oscillators in one-to-one correspondence with the unperturbed modes ψ_n. Although the left-hand side of (4.36) describes the motion of a linear oscillator, the forcing function F_N will in general contain terms in η_n representing linear and nonlinear damping, springiness and inertia. Consequently, it is easy to find familiar nonlinear differential equations as special cases of (4.36). Such special results aid

greatly the interpretations of complicated observed behavior in terms of simpler elementary motions. Thus it is important to understand the connections between parameters defining the oscillators, the characteristics of the modes, and the definitions provided in the process of spatial averaging.

(v) Different problems are distinguished chiefly in two respects: Geometry of the combustor; and the form of the forcing function F_N. The forcing function contains the influences of gasdynamics explicitly, but all other processes must be modeled, either with theory or based on experimental results. The geometry and the boundary conditions determine the modal expansion functions ψ_n and the frequencies ω_n. For complicated geometries, as for many large solid propellant rockets and for most gas turbine combustors, computation of the ψ_n and ω_n has been a time-consuming and expensive process. That situation is gradually changing with the development of more capable software (e.g., French 2003; French and Flandro 2003).

(vi) The relatively general context in which the oscillator equations (4.36) have been derived does not exclude simpler problems which can either be treated as special cases or constructed without reference to the procedures worked out here. (See, e.g., Section 2.6) However, it is then often more difficult to be certain that all important processes are accounted for or properly ignored.

4.4. An Alternative Application of Spatial Averaging: The Method of Least Residuals

With a series of works beginning in the late 1960s, Professor B.T. Zinn and his students developed and applied a different method based on spatial averaging, an interesting extension of Galerkin's method. See Powell (1970); Powell and Zinn (1969; 1971a,b; 1974); Zinn and Powell (1968; 1970); Lores and Zinn (1973). There are necessarily some similarities with the method discussed in the preceding two sections; in particular the formal results should agree in detail, or at least be reconcilable. There are, however, distinct differences both in the sequence of historical developments and in many important matters of applications.

In respect to the historical developments, the Georgia Tech group was first to apply spatial averaging to analyze nonlinear behavior, in liquid propellant rockets. They were also first to demonstrate several nonlinear phenomena confirmed later in works published by others. Those results are reviewed here in Chapter 7. It was three years after the first Georgia Tech report that Culick independently worked out a much simplified form of the method described in Section 4.2, to explain nonlinear behavior observed in a laboratory device, the T-burner, used to measure the combustion dynamics of solid propellants. At that time, there was practically no communication (or mutual attention) between the liquid and solid rocket communities at the research level, a condition that blocked certain benefits, but which has since been corrected.

Recently, Seywert and Culick (1998) showed that when applied to the same equation, the two methods lead to the same formal result (4.28). To establish that conclusion, we follow the formal procedure discussed by Finlaysen and Scriven (1966) as extended by Powell (1970); Powell and Zinn (1969, 1971a) and Zinn and Powell (1968, 1970) to account for the inhomogeneous boundary conditions. The method is in fact quite general, capable of handling much more elaborate problems than that for which the method covered in Section 4.3 has been worked out. Here we take a direct route to make the main point most clearly. Write the wave equation (4.1) and its boundary condition as

$$\begin{aligned} E(\mathbf{r},t) &= 0 \\ B(\mathbf{r},t) &= 0 \end{aligned} \qquad (4.37)\text{a,b}$$

where

$$\begin{aligned} E(p') &:= \nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} - h \\ B(p') &:= \hat{\mathbf{n}} \cdot \nabla p' + f \end{aligned} \qquad (4.38)\text{a,b}$$

The Galerkin method is based on expansion of the dependent variables in a set of basis functions ϕ_n. Here we suppose that there is only the single variable p' in the problem[9] defined by (4.37). The approximation $\tilde{p}$ expressed as the expansion truncated to M terms is assumed to become equal to the solution p as $M \to \infty$:

$$\lim_{M\to\infty} \tilde{p} = \lim_{M\to\infty} \sum_{m=1}^{M} a_m \phi_m = p \tag{4.39}$$

At this stage the functions ϕ_n are undefined. The modified Galerkin method consists in spatially averaging, with a weighting function, the governing equation and its boundary condition, both applied to an approximation $\tilde{p}$, and requiring that the difference vanish:

$$\iiint_V E(\tilde{p})\phi_N dV - \oiint_S B(\tilde{p})\phi_N dS = 0 \tag{4.40}$$

The spatial weighting function ϕ_N need not be one of the basis functions, but we made that choice to establish the equivalence of the methods without unnecessary complications.

Now substitute the definition (4.36) for E and B:

$$\iiint_V \left(\nabla^2\tilde{p}' - \frac{1}{\bar{a}^2}\frac{\partial^2\tilde{p}'}{\partial t^2}\right)\phi_N dV - \oiint_S \left(\hat{\mathbf{n}}\cdot\nabla\tilde{p}' + f\right)\phi_N dS = 0 \tag{4.41}$$

Apply Green's theorem to the first term to give

$$\iiint_V \nabla^2\tilde{p}'\phi_N dV = \oiint_S \left(\tilde{p}'\nabla\phi_N - \phi_N\nabla\tilde{p}'\right)\cdot\hat{\mathbf{n}}\, dS - \iiint_V \tilde{p}'\nabla^2\phi_N dV$$

and choose $\phi_N \equiv \psi_N$ defined by (4.12):

$$\iiint_V \nabla^2\tilde{p}'\psi_N dV = -\oiint_S \psi_N\nabla\tilde{p}'\cdot\hat{\mathbf{n}}\, dS + k_N^2 \iiint_V \tilde{p}'\psi_N dV$$

Substitution in (4.41), with $\phi_N \equiv \psi_N$ and some rearrangement leads to the result identical to (4.30):

$$\iiint_V \frac{1}{\bar{a}^2}\frac{\partial^2\tilde{p}'}{\partial t^2}\psi_N dV + k_N^2 \iiint_V \tilde{p}'\psi_N dV = -\left\{\iiint_V \psi_N h dV + \oiint_S \psi_N f dS\right\} \tag{4.42}$$

Owing to the care taken to recognize the approximation to p' with a truncated expansion, $\tilde{p}'$ appears here in place of p' in (4.30).

The preceding remarks establish the equivalence of the methods only for the case when the equations of motion are written for the fluctuations of the flow variable and then combined to form the wave equation for the pressure. It should be apparent from the discussion in Sections 4.1–4.3 that the method developed there, and used throughout the remainder of this book, is restricted to that formulation. In contrast, the Galerkin method is not constrained to any particular form of the governing equations; of course the problem to be analyzed must lend itself to definition of basis functions. In that sense, the modified Galerkin method is potentially more general than the method discussed in Sections 4.1–4.3 and used throughout this book. However, the method has not been extended beyond the applications investigated by the group at Georgia Tech many years ago and for several reasons seems not to have motivated others to pursue even similar applications. In almost all their work, the Georgia Tech group introduced a form of a potential for the unsteady velocity. Combined with sometimes vague usage of expansion parameters and ordering, that practice renders the method awkward to use and the results difficult to interpret.

[9]The other variables of the flow field must also be expanded, the procedure being the same as that followed in the method based on the results of Section 4.3.

4.5. Application of Time-Averaging

To this point the expansion procedure based on two small parameters has been used only to derive the systems of equations describing successively more difficult classes of problems listed in Section 3.3.2. Later we will see how an iterative method based partly on the expansion reduces those systems of equations to more readily soluble forms. In this section we apply time-averaging to convert the second-order equations (4.36) to first order equations. First, two remarks:

(i) Use of time-averaging is motivated by the experimental observation that combustion instabilities commonly show slowly varying amplitudes and phases of the modes contributing to the motions. That behavior is a consequence of the relative weakness of the disturbing processes and is therefore measured by the small parameter μ characteristic of the Mach number of the mean flow. It is essential to understand that it is not the amplitudes themselves (i.e. the parameter ε) that matters. Thus the application of time-averaging in the present context is *not* intended to treat nonlinear behavior in μ, but is based on the idea that there is only weak coupling between the mean flow and the unsteady motions, proportional to the Mach number of the average flow. Nonlinear behavior of higher order in ε is a distinct matter, formally unaffected by the time-averaging.

(ii) Two-time scaling (Kevorkian and Cole, 1996; Cole 1968) is an alternative method to time-averaging. The results obtained are identical up to second order acoustics (Sections 3.3.3(II) and 3.4), a conclusion not proved here but consistent with similar previous works in other fields.

According to the discussion in Section 3.3.3, we can characterize the functions h and f, and hence the forcing function F_N, as sums of terms each of which is of order μ and of zeroth or first order in ε. Thus if we reactivate the ordering parameters μ and ε, the right-hand side of (3.45)b, for example, has the form

$$-\mu\varepsilon\left\{\{[\mathbf{M}]\}_1 + \frac{\varepsilon}{\mu}\{\mathbf{M}\}_2\right\} + \boldsymbol{\mathcal{F}'}$$

The divergence of these terms eventually appears in h and F_n. Hence we are justified in taking F_n of order μ; to show this explicitly write (4.36) as

$$\ddot{\eta}_N + \omega_N^2\eta_N = \mu G_N \tag{4.43}$$

In any event, for μ small, the η_N differ but little from sinusoids so (without approximation) it is reasonable to express $\eta_N(t)$ in the equivalent forms

$$\eta_N(t) = r_N(t)\sin\left(\omega_N t + \phi_N(t)\right) = A_N(t)\sin\omega_N t + B_N(t)\cos\omega_N t \tag{4.44}$$

and

$$\begin{gathered} A_N(t) = r_N\cos\phi_N \ ; \quad B_N = r_N\sin\phi_N \\ r_N = \sqrt{A_N^2 + B_N^2} \ ; \quad \phi_N = \tan^{-1}\left(\frac{A_N}{B_N}\right) \end{gathered} \tag{4.45}$$

One way to proceed follows a physical argument based on examining the time evolution of the energy of the oscillator having amplitude η_N (Culick 1975). The energy $\mathcal{E}_N$ is the sum of kinetic and potential energies,

$$\mathcal{E}_N(t) = \frac{1}{2}\dot{\eta}_N^2 + \frac{1}{2}\omega_N^2\eta_N^2 \tag{4.46}$$

The time-averaged values of the energy and power input to the oscillator, due to the action of the force μG_N, are

$$\langle\mathcal{E}_N\rangle = \frac{1}{\tau}\int_t^{t+\tau}\mathcal{E}_N dt' \ ; \quad \langle\mu G_N\dot{\eta}_N\rangle = \frac{1}{\tau}\int_t^{t+\tau}\mu G_N\dot{\eta}_N dt' \tag{4.47}$$

Conservation of energy requires that the time-averaged rate of change of energy equal the time-averaged rate of work done by μG_N on the oscillator:

$$\frac{d}{dt}\langle \mathcal{E}_N \rangle = \mu \langle G_N \dot{\eta}_N \rangle \tag{4.48}$$

From (4.44), the velocity is

$$\dot{\eta}_N = \omega_N r_N \cos(\omega_N t + \phi_N) + \left[\dot{r}_N \sin(\omega_N t + \phi_N) + \dot{\phi}_N r_N \cos(\omega_N t + \phi_N)\right] \tag{4.49}$$

Following Krylov and Bogoliubov (1947) we apply the 'strong' condition that the velocity is always given by the formula for an oscillator is force-free-motion,

$$\dot{\eta}_N = \omega_N r_N \cos(\omega_N t + \phi_N) \tag{4.50}$$

Hence (4.49) is consistent with this requirement only if

$$\dot{r}_N \sin(\omega_N t + \phi_N) + \dot{\phi}_N r_N \cos(\omega_N t + \phi_N) = 0 \tag{4.51}$$

Now use the definitions (4.44), (4.46) and (4.50) to find

$$\mathcal{E}_N = \frac{1}{2}\omega_N^2 r_N^2$$

$$\mu G_N \dot{\eta}_N = \mu G_N \omega_N r_N \cos(\omega_N t + \phi_N) \tag{4.52)a,b}$$

The statement "slowly varying amplitude and phase" means that the fractional changes of amplitude and phase are small in one cycle of the oscillation and hence during the interval of averaging τ, if τ is at least equal to the period of the fundamental mode:

$$\frac{\tau}{r_N}\frac{dr_N}{dt} \ll 1 \; ; \quad \frac{\tau}{2\pi}\frac{d\phi_N}{dt} \ll 1 \tag{4.53}$$

These inequalities imply that r_N and ϕ_N may be treated as constants during the averaging carried out in (4.47). To see this, imagine that r_N for example, is expanded in Taylor series for some time t_1 in the interval τ, $t < t_1 < t + \tau$:

$$r_N(t) = r_N(t_1) + (t - t_1)\left(\frac{dr_N}{dt}\right)_{t_1} + \cdots$$

For r_N slowly varying, $\dot{r}_N$ doesn't vary much during a period and may be assigned some average value. The increment $t - t_1$ has maximum value τ; so the second term is negligible according to the first of (4.47). Therefore $r_N(t) \approx r_N(t_1)$ for any t_1 in the interval of averaging and the assertion is proved.

Substitution of (4.52)b in (4.48) then gives

$$\omega_N^2 r_N \frac{dr_N}{dt} = \mu \frac{\omega_N r_N}{\tau} \int_t^{t+\tau} G_N \cos(\omega_N t' + \phi_N) dt'$$

which gives

$$\frac{dr_N}{dt} = \mu \frac{1}{\omega_N \tau} \int_t^{t+\tau} G_N \cos(\omega_N t' + \phi_N) dt' \tag{4.54}$$

Because $r_N(t)$ and $\dot{r}_N(t)$ are nearly constant in the interval $(t, t+\tau)$, this relation implies the equation *before* averaging,

$$\frac{dr_N}{dt} = \frac{\mu}{\omega_N} G_N(t) \cos(\omega_N t + \phi_N) \tag{4.55}$$

The corresponding equation for the phase $\phi_N(t)$ before averaging is found by first rearranging (4.51) to give

$$r_N \frac{d\phi_N}{dt} \cos(\omega_N t + \phi_N) = -\dot{r}_N \sin(\omega_N t + \phi_N)$$

which becomes, after substitution of (4.55),

$$r_N \frac{d\phi_N}{dt} = -\frac{\mu}{\omega_N} G_N(t) \sin(\omega_N t + \phi_N) \tag{4.56}$$

Now time average this equation over the interval τ; the left-hand side is approximately constant for theorem give above, and the equation for $\phi_N(t)$ is

$$r_N \frac{d\phi_N}{dt} = -\mu \frac{1}{\omega_N \tau} \int_t^{t+\tau} G_N \sin(\omega_N t' + \phi_N) dt' \tag{4.57}$$

With the relations (4.45), equations (4.54) and (4.57) can be converted to equations for A_N and B_N:

$$\frac{dA_N}{dt} = \frac{\mu}{\omega_N \tau} \int_t^{t+\tau} G_N \cos \omega_N t' dt'$$

$$\frac{dB_N}{dt} = -\frac{\mu}{\omega_N t} \int_t^{t+\tau} G_N \sin \omega_N t' dt' \qquad (4.58)\text{a,b}$$

Whichever pair one chooses to use, (4.54) and (4.57) or (4.55), the general formal problem of solving a system of coupled second order equations (4.43) for the oscillators, has been converted to the simpler approximate formal problem of solving a system of coupled first order equations. The essential basis for that conversion is the removal of the fast oscillatory behavior with the definition (4.44), a transformation made reasonable because the changes of amplitudes and phases take place on a much slower (i.e. longer) time scale than do the oscillations. The presence and role of two time scales is more evident in the following alternative derivation.

From the second equality of (4.44), we find the velocity

$$\dot{\eta}_N = \omega_N \left[A_N \cos \omega_N t - B_N \sin \omega_N t\right] + \left[\dot{A}_N \sin \omega_N t + \dot{B}_N \cos \omega_N t\right]$$

Now enforce the condition corresponding to (4.51),

$$\dot{A}_N \sin \omega_N t + \dot{B}_N \cos \omega_N t = 0 \tag{4.59}$$

and the velocity is

$$\dot{\eta}_N = \omega_N \left[A_N \cos \omega_N t - B_N \sin \omega_N t\right] \tag{4.60}$$

Substitution in (4.43) gives

$$\omega_N \left[\dot{A}_N \cos \omega_N t - \dot{B}_N \sin \omega_N t\right] + \omega_N^2 \left[-A_N \sin \omega_N t - B_N \cos \omega_N t\right] + \omega_N^2 \left[A_N \sin \omega_N t + B_N \cos \omega_N t\right] = \mu G_N$$

and

$$\dot{A_N} \cos \omega_N t - \dot{B_N} \sin \omega_N t = \frac{\mu}{\omega_N} G_N$$

Multiply by $\cos \omega_N t$ and substitute (4.59) for $\dot{B}_N \cos \omega_N t$ to give

$$\dot{A_N} \cos^2 \omega_N t - \sin \omega_N t \left[-\dot{A_N} \sin \omega_N t \right] = \frac{\mu}{\omega_N} G_N \cos \omega_N t$$

so

$$\frac{dA_N}{dt} = \frac{\mu}{\omega_N} G_N \cos \omega_N t \tag{4.61}$$

Similarly,

$$\frac{dB_N}{dt} = -\frac{\mu}{\omega_N} G_N \sin \omega_N t \tag{4.62}$$

Now introduce two time-scales, τ_f the first scale, of the order of the period of the fundamental oscillation (in fact, we might as well set $\tau_f = 2\pi/\omega_1$; and τ_s, the slow scale characterizing transient changes of the amplitudes and phases of the oscillations. Two corresponding dimensionless time variables can be defined, $t_f = t/\tau_f$ and $t_s = t/\tau_s$. Thus we consider the amplitudes and phases to be functions of the slow variable t_s while the forcing functions G_N depend on both t_f and to because they depend on the η_N, $(i = 1, 2, \dots)$

$$\eta_N = A_N(t_s) \sin \left(2\pi \frac{\omega_N}{\omega} t_f \right) + B_N(t_s) \cos \left(2\pi \frac{\omega_N}{\omega_1} t_f \right) \tag{4.63}$$

In terms of the dimensionless time variables, is

$$\frac{1}{\tau_s} \frac{dA_N}{dt_s} = \frac{\mu}{\omega_N} G_N \cos \omega_N t$$

and averaging over the fast variable we have

$$\frac{1}{\tau_s} \int_{t_f}^{t_f+\tau_f} \frac{1}{\tau_s} \frac{dA_N}{dt'_s} dt'_f = \frac{\mu}{\omega_N} \frac{1}{\tau_f} \int_{t_f}^{t+\tau_f} G_N \cos \left(2\pi \frac{\omega_N}{\omega_1} t'_f \right) dt'_f$$

On the left-hand side, dA_N/dt'_s is assume to be sensibly constant in the interval τ_f and we have

$$\frac{1}{\tau_s} \frac{dA_N}{dt'_s} = \frac{\mu}{\omega_N \tau_f} \int_{t_f}^{t+\tau_f} G_N \left(t'_f, t'_s \right) \cos \left(2\pi \frac{\omega_N}{\omega_1} t'_f \right) dt'_f \tag{4.64}$$

Those parts of G_N depending on t'_s are taken also to be constant and if we now rewrite this equation in terms of dimensional variables, we recover (4.58)a with $\tau = \tau_f = 2\pi/\omega$. Similar calculations will produce again (4.58)a. Note that due to the nonlinear coupling, the amplitude and phases of all modes normally change on roughly the same scale as that for the fundamental mode; thus the single interval of averaging works for all modes.

Krylov and Bogoliubov (1947) discuss procedures for carrying the results of time-averaging to higher order in the small parameter characterizing the expansion. Here, however, the parameter is μ, a Mach number characteristic of the mean flow. Hence, as emphasized in Chapter 3, **extension to higher order in μ is not justified without re-deriving the basic equations to account for higher orders in the mean flow from the beginning**.

With μG_N replaced by F_N, letting $N \to n$, and $\tau_s t_s = t$, we find the usual forms of the time-averaged equations,

$$\frac{dA_n}{dt} = \frac{1}{\omega_n \tau} \int_t^{t+\tau} F_n \cos \omega_n t' dt'$$

$$\frac{dB_n}{dt} = -\frac{1}{\omega_n \tau} \int_t^{t+\tau} F_n \sin \omega_n t' dt' \qquad (4.65)\text{a,b}$$

Inserting the definitions of t_s and t_f in (4.63), and replacing N by n, we have the expression for $\eta_n(t)$:

$$\eta_n(t) = A_n(t) \sin(\omega_n t) + B_n(t) \cos(\omega_n t) \qquad (4.66)$$

In Chapter 7 we will use a continuation method to assess the ranges of parameters and other conditions for which the first order equations give accurate results when compared with solutions to the complete oscillator equations. In the development of the theoretical matters described in this book, the sets of first order equations, (4.65)a,b, have been central. They remain extremely useful both for theoretical work and for applications.

4.6. The Procedure for Iterative Solution to the Oscillator Equations

The oscillator equations (4.35) and (4.36) are not yet in a form that can be readily solved because the functions F_N, defined by (4.32) contain not only p' but also the dependent variables ρ', T' and $\mathbf{u}'$ in the functions h and f. With the two-parameter expansion as the basis, the iteration procedure provides a means for expressing F_N in terms of p' only to first order in ε. Thus eventually the oscillator equations become a system soluble for the modal amplitudes $\eta_N(t)$. There are of course approximations required, but the orders of their effects can always be estimated in terms of the parameters ε and μ. To appreciate how the procedure is constructed, it is helpful always to keep in mind the correspondence between the smallness of ε and μ, and the distortions they represent of the unperturbed classical acoustic field.

There are two classes of distortions or perturbations: Those represented by higher orders of ε, arising as nonlinear effects of finite amplitudes,[10] classified generally as energy transfer between modes; and those measured by μ, consequences of interactions, hence energy transfer, between the steady and unsteady fields. Each of those types of perturbations may be identified within the volume in question and at the boundary. Quite generally, then, we must take into account perturbations of the classical acoustic field, associated with three kinds of energy transfer: Linear transfer between the mean and fluctuating motions; nonlinear transfer between modes, or mode coupling, independent of the average flow field; and nonlinear energy transfer between the mean flow and fluctuating fields. Those three kinds of energy transfer characterize, respectively, the Problems I; II; III; and IV defined in Section 3.3. The way in which we view and accommodate those perturbations determines our choice of basis functions ψ_N used in the modal expansion[11] (4.27). In this work we are not accounting for nonlinear energy transfer between the mean flow and fluctuating fields; i.e. those contributions represented by terms $O(\mu\varepsilon^2)$, are ignored as in Section 3.4.

[10] Recall that in this work, nonlinear behavior is measured in terms of the amplitude ε of the unsteady motions. It is intrinsic to their derivation (Chapter 3) that the governing equations are linear in μ, i.e. in the Mach number of the mean flow. Hence, with those equations, expansions and solutions cannot legitimately be carried further then first order in μ. The procedures developed here can be carried formally to higher order in ε.

[11] The expansion is not adequate for treating Problem IV defined in Section 3.3. In that case the basis functions must include distortions of order μ.

4.6.1. Iteration on the Mach Number of the Mean Flow. To simplify the discussion here we assume that the functions h and f contain only four sorts of terms: small, roughly of order ε, but not strictly ordered because they are independent of the fluctuating field; linear in the fluctuations, and hence of order ε; bilinear in the mean flow speed and the fluctuations, and hence of order $\varepsilon\mu$; and nonlinear terms in the fluctuations of order ε^2 and ε^3. The terms are additive, so the two functions can be written formally by rearranging (3.63)a,b in the form

$$\begin{aligned} h &= \varepsilon(h_{00} + h_{10}) + \varepsilon\mu h_{11} + \varepsilon^2 h_{20} + \varepsilon^3 h_{30} \\ f &= \varepsilon(f_{00} + f_{10}) + \varepsilon\mu f_{11} + \varepsilon^2 f_{20} + \varepsilon^3 f_{30} \end{aligned} \qquad (4.67)\text{a,b}$$

where h_{00} and f_{00} do not depend on the unsteady field and hence constitute true forcing functions.

The terms h_{10} and f_{10} are linear in the fluctuations and independent of the mean flow speed. They can be immediately combined with the wave operator on the left-hand side of the oscillator equations, showing that they represent attenuation and frequency shifts of classical acoustic modes. A particularly clear and important example is the viscous acoustic boundary layer on a rigid impermeable wall discussed in Annex C and Section 5.9.

Almost all of the processes responsible for linear instabilities in a combustion system are contained in the functions h_{11} and f_{11} with possible contributions from h_{10} and f_{10}. At this stage, it is not clear what phenomena might be represented by the quadratic and cubic nonlinearities not included here. Detailed investigations are required to address the question.

The special reason for retaining those terms shown in (4.67)a,b is the following. Within the ordering procedure followed here, only the unperturbed classical modes ψ_n are required to generate explicit equations for the modal amplitudes with the oscillator equations (4.36). To see the problem, substitute (4.67)a in the right-hand side of the wave equation (4.1)a for p':

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = \varepsilon(h_{00} + h_{10}) + \varepsilon\mu h_{11} + \varepsilon^2 h_{20} + \varepsilon^3 h_{30} \qquad (4.68)$$

The functions h_0, h_{10}, h_{11}, ... are found by identification with (3.58) and the definitions of the brackets given in Section A.6. While h_{11}, h_{20} and h_{30} can be found explicitly with those formulas, h_{00} and h_{10} arise from $\boldsymbol{\mathcal{F}}'$ and $\mathcal{P}'$ and are determined by the models of processes other than the fluid mechanics covered by the compressible Euler equations. The acoustic boundary layer, for example, contributes to h_{10} and f_{10}; see Section 6.5 for another example.

In the iterative procedure, the parameter μ defines the iterations so that nonlinear behavior governed by ε is present at each stage to the order selected. The zeroth approximation is the classical acoustic field given by the modal expansions with time dependent amplitudes:

$$\begin{aligned} p'_O &= \bar{p}\sum_0^M \eta_m(t)\psi_m(\mathbf{r}) \\ \mathbf{u}'_O &= \bar{a}^2 \sum_0^M \frac{\dot{\eta}_m(t)}{\gamma k_m^2}\nabla\psi_m(\mathbf{r}) \end{aligned} \qquad (4.69)\text{a,b}$$

The expansion for the velocity field has been chosen so that term by term the expansions satisfy the classical acoustic momentum equation free of perturbations

$$\bar{\rho}\frac{\partial \mathbf{u}'_O}{\partial t} + \nabla p'_O = 0 \qquad (4.70)$$

Correspondingly, the values of density and temperature fluctuations are set by the adiabatic relations,

$$\frac{\rho}{\bar{\rho}} = \left(\frac{p}{\bar{p}}\right)^{1/\gamma} \quad , \quad \frac{T}{\bar{T}} = \left(\frac{p}{\bar{p}}\right)^{\frac{\gamma-1}{\gamma}}$$

Set $p = \bar{p} + p'$, $\rho = \bar{\rho} + \rho'$, $T = \bar{T} + T'$ and expand to find to second order

$$\begin{aligned} \frac{\rho'_O}{\bar{\rho}} &= \frac{1}{\gamma}\frac{p'_O}{\bar{p}} - \frac{\gamma-1}{2\gamma^2}\left(\frac{p'_O}{\bar{p}}\right)^2 \\ \frac{T'_O}{\bar{T}} &= \frac{\gamma-1}{\gamma}\frac{p'_O}{\bar{p}} - \frac{\gamma-1}{2\gamma^2}\left(\frac{p'_O}{\bar{p}}\right)^2 \end{aligned} \qquad (4.71)\text{a,b}$$

Then the equation for the first approximation, $p'^{(1)}$, to p' is found by setting the fluctuations of flow variables in h equal to their classical values and retaining only terms of appropriate order. To see how this works, begin with (2.78)a,b. Define the function $\mathcal{G}$

$$\mathcal{G} = \bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\boldsymbol{\mathcal{F}'}_{III} - \frac{1}{\bar{a}^2}\frac{\partial \mathcal{P}'_{III}}{\partial t} \qquad (4.72)$$

The form of $\mathcal{G}$ is unspecified, but we assume that the processes represented lead to terms of all orders retained in the problem at hand. Then with the ordering parameters shown and the acoustic approximations (4.69)a,b substituted for all fluctuations, indicated by the subscript $(\quad)_O$, (3.63)a,b give[12]

$$\begin{aligned} h := h_{III} = \varepsilon\left[\mathcal{G}_{00} + \mathcal{G}_{10}\right]_O &+ \varepsilon\mu\left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{[\mathbf{M}]\}_1 + \frac{1}{\bar{a}^2}\frac{\partial\{[p]\}_1}{\partial t} + \frac{1}{\bar{\rho}}\nabla\bar{\rho}\cdot\nabla p' + \mathcal{G}_{11}\right]_O \\ &+ \varepsilon^2\left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{\mathbf{M}\}_2 + \frac{1}{\bar{a}^2}\frac{\partial\{p\}_2}{\partial t} + \mathcal{G}_{20}\right]_O \\ &+ \varepsilon^3\left[-\bar{\rho}\nabla\cdot\frac{1}{\bar{\rho}}\{\mathbf{M}\}_3 + \mathcal{G}_{30}\right]_O \end{aligned} \qquad (4.73)$$

$$\begin{aligned} f := f_{III} = \varepsilon\left[\bar{\rho}\left(\frac{\partial\mathbf{M}'}{\partial t}\right)_{10} - (\boldsymbol{\mathcal{F}'})_{10}\right]_O\cdot\hat{n} &+ \varepsilon\mu\left[\bar{\rho}\left(\frac{\partial\mathbf{M}'}{\partial t}\right)_{11} + \{[\mathbf{M}]\}_1 - (\boldsymbol{\mathcal{F}'})_{11}\right]_O\cdot\hat{n} \\ + \varepsilon^2\left[\bar{\rho}\left(\frac{\partial\mathbf{M}'}{\partial t}\right)_{20} + \{\mathbf{M}\}_2 - (\boldsymbol{\mathcal{F}'})_{20}\right]_O &+ \varepsilon^3\left[\bar{\rho}\left(\frac{\partial\mathbf{M}'}{\partial t}\right)_{30} + \{\mathbf{M}\}_3 - (\boldsymbol{\mathcal{F}'})_{30}\right]_O \end{aligned} \qquad (4.74)$$

Substitution of 4.73 and 4.74 into the oscillator equations (4.36),

$$\ddot{\eta}_N + \omega_N^2\eta_N = -\frac{\bar{a}_r^2}{\bar{p}_r E_N^2}\left\{\iiint_V h_{III}\psi_N dV + \oiint_S f_{III}\psi_N dS\right\} \qquad (4.75)$$

produces the equations for the amplitudes of the unperturbed modes.

Because the basis functions ψ_m satisfy the boundary conditions for the unperturbed problem, $p'_O(r,t)$ given by (4.69)a, also does not satisfy the actual boundary conditions—it is the zeroth approximation. It is the first order approximation to the pressure field that satisfies the correct boundary conditions, which are first order in μ. Similarly, the unsteady velocity to zeroth order is given by the expansion (4.69)b and not only doesn't satisfy the actual boundary conditions but is clearly an irrotational field, $\nabla\times u'_O = 0$.

[12]Second and third order terms are shown for completeness in (4.73) and (4.74); they will not be needed until the discussion of nonlinear behavior in Chapter 7.

4.6.2. Zeroth and First Order Solutions for the Eigenvalues and Basis Functions. On several occasions, researchers in this field, and users of results, have seriously misunderstood the meaning of the preceding procedure. Its purpose is to give the **first order** (in μ) results for the time-evolution of modal amplitudes (which to zeroth order are constant in the scheme outlined here) and the **first order** (in μ) formulas for the eigenvalues of steady oscillations when the perturbations are accounted for in **first order** (in ε) i.e. linear stability. The complex eigenvalues for steady waves are found by setting[13]

$$\eta_N = \hat{\eta}_N e^{-i\bar{a}kt}$$

$$h = \hat{h} e^{-i\bar{a}kt} \qquad (4.76)\text{a,b}$$

$$f = \hat{f} e^{-i\bar{a}kt}$$

Substitution in (4.75) gives

$$\hat{\eta}_N \left(-\bar{a}^2 k^2 + \omega_N^2\right) = -\frac{\bar{a}^2}{\bar{p} E_N^2} \left\{ \iiint_V \hat{h} \psi_N dV + \oiint_S \hat{f} \psi_N dS \right\} \qquad (4.77)$$

Equation (4.77) is useful only if h and f are linear in the fluctuations, for then every term has $e^{-i\bar{a}kt}$ as a factor. Hence from (4.67)a,b only the terms h_{10}, h_{11}, f_{10} and f_{11} can be retained. The formula (4.77) for the eigenvalues of the perturbed (actual) problem to first order is

$$\begin{aligned}(\bar{a}k)^2 = \omega_N^2 + \frac{\bar{a}^2}{\bar{p} E_N^2} \Bigg\{ & \varepsilon \left[\iiint_V \hat{h}_{10} \psi_N dV + \oiint_S \hat{f}_{10} \psi_N dS \right] \\ & + \varepsilon\mu \left[\iiint_V \hat{h}_{11} \psi_N dV + \oiint_S \hat{f}_{10} \psi_N dS \right] \Bigg\}\end{aligned} \qquad (4.78)$$

This is essentially the result used widely for examining the linear stability of combustion chambers, the subject of Chapter 6.

Now the question is—how can the eigenfunctions, the mode shapes, corresponding to (4.78) be computed. In fact the first order modes have never been computed and are not required if one is satisfied with perturbations to the order carried in (4.78). We return to (4.68) written now for steady waves, with the exponential time dependence; so $p' = \hat{p} e^{-i\bar{a}kt}, \ldots$. We also drop the terms h_{00}, h_{20} and h_{30}, giving the equation for $\hat{p}_1$:

$$\nabla^2 \hat{p}^{(1)} + k^2 \hat{p}^{(1)} = \varepsilon \hat{h}_{10} + \varepsilon\mu \hat{h}_{11} \qquad (4.79)$$

Continuing the iteration procedure, the eigenfunction of zeroth order, ψ_n and its gradient $\nabla\psi_n$ are substituted in the right-hand side wherever p' and $\mathbf{u}'$ appear. Hence (4.79) becomes an inhomogeneous equation with the right-hand side given, say $\mathcal{H}(\mathbf{r})$:

$$\nabla^2 \hat{p}^{(1)} + k^2 \hat{p}^{(1)} = \mathcal{H} \qquad (4.80)$$

with

$$\mathcal{H} = \varepsilon \hat{h}_{10} + \varepsilon\mu \hat{h}_{11} \qquad (4.81)$$

[13]Here $\bar{a}k$ is the complex frequency in the actual motion, i.e. containing perturbations to first order from the basis (modal) functions.

This equation is most conveniently solved using the Green's function defined in Section 4.1, leading to the solution (4.20), here having the form

$$\hat{p}^{(1)}(\mathbf{r}) = \psi_N(\mathbf{r}) + \sum_{n=0}^{\infty}{}' \frac{\psi_n(\mathbf{r})}{E_n^2\left(k^2 - k_n^2\right)} \left\{ \iiint_V \psi_n \mathcal{H} dV + \oiint_S \psi_n \mathcal{F}_n dS \right\} \tag{4.82}$$

where $\mathcal{F}_n$ is the boundary condition extracted from (4.67)b consistent with the manner in which (4.67)a was handled,

$$\mathcal{F}_n = \varepsilon \hat{f}_{10} + \varepsilon\mu \hat{f}_{11} \tag{4.83}$$

Although (4.82) seems to satisfy the same boundary conditions as the classical mode shape—because each term contains ψ_n—that is not the case. The reason has to do with the behavior of the infinite series representing the Green's function, equation (4.15). Near the boundary, the function behaves as a delta-function and (4.82) does satisfy the correct boundary condition. The result is not proved here; a good discussion of the matter may be found in Morse and Feshbach (1952). See Annex F for a simpler example.

Finally, the velocity field must be computed to first order by using the linear form of the momentum equation (3.47)b:

$$\bar{\rho}\frac{\partial \mathbf{M}'}{\partial t} + \nabla p' = -\{[\mathbf{M}]\}_1 + \boldsymbol{\mathcal{F}}' \tag{4.84}$$

Again the zeroth order acoustic values are substituted in the right-hand side, and for steady waves, with the mode shape for the pressure given by (4.82), we have for steady waves

$$i\bar{\rho}\bar{a}k\mathbf{M}'^{(1)} = -\nabla\hat{p}^{(1)} - \varepsilon\mu\{[\hat{\mathbf{M}}]\}_1 + \varepsilon\hat{\boldsymbol{\mathcal{F}}}_{10} + \varepsilon\mu\hat{\boldsymbol{\mathcal{F}}}_{11} \tag{4.85}$$

It is particularly important to notice that **this first order field is in general not irrotational**, possibly the most commonly misunderstood result of the entire procedure developed here. Even though the basis functions used in the zeroth order modal expansion expressed an irrotational field, the field computed to first order may be rotational. Put another way, the approximate procedure proposed and advocated here really does have the potential for handling approximately a wide variety of realistic problems with minimal difficulty. The unspecified function $\boldsymbol{\mathcal{F}}'$ is key in that respect. One has virtually complete freedom in choosing that function. Especially, $\boldsymbol{\mathcal{F}}'$ can be constructed so that $\mathbf{M}'$ satisfies a given boundary condition. For example, one may require that the velocity should be normal to the boundary when there is incoming flow, generally accepted to be the case for a burning solid propellant (Flandro 1995a). This matter will be discussed further in Section 6.9, where some aspects of the 'flow-turning problem' are examined.