ANNEX A

Equations of Motion

Combustion systems commonly contain condensed phases: liquid fuel or oxidizer and combustion products including soot and condensed metal oxides. Hence the equations of motion must account for two or three phases and at least one species in each. For investigating the dynamics of combustors, it is entirely adequate to represent each phase as its mass average over all member species. It is unnecessary to distinguish liquid and solid material and we assume a single species in the condensed phase, devoted by subscript $(\quad)_l$. For some applications it is appropriate to extend the representation slightly to accommodate distributions of particle sizes, not included in this annex. There is some advantage to treating the gas phase as a multi-component reacting mixture. As the primitive conservation equations we therefore begin with the following set:

A.1. General Equations of Motion

Conservation of Species, Gas Phase[1]

$$\frac{\partial \rho_{gi}}{\partial t} + \nabla \cdot (\rho_{gi}\mathbf{u}_{gi}) = w_{gi} + w_{gi}^{(l)} + w_{gei} \tag{A.1}$$

Global Conservation of Mass, Gas Phase

$$\frac{\partial \rho_g}{\partial t} + \nabla \cdot (\rho_g\mathbf{u}_g) = w_g^{(l)} + w_{eg} \tag{A.2}$$

Global Conservation of Mass, Condensed Phase[2]

$$\frac{\partial \rho_l}{\partial t} + \nabla \cdot (\rho_l\mathbf{u}_l) = -w_g^{(l)} + w_{el} \tag{A.3}$$

Global Conservation of Momentum

[1]Superscript $(\quad)^{(l)}$ on $w_{gi}^{(l)}$ means that the source material is liquid. Thus $w_{gi}^{(l)}$ is the rate at which species i of the gas phase is produced from the liquid phase. The dimensions of $w_{gi}^{(l)}$ are mass per unit volume of space, per unit time.

[2]Note that ρ_l represents the mass of condensed material per unit volume of chamber, *not* the density of the material itself. Subscript $(\quad)_l$ denotes 'liquid'; at the temperature prevailing in combustion chambers, most condensed materials are liquid.

$$\frac{\partial}{\partial t}\left(\sum^{i}\rho_{gi}\mathbf{u}_{gi}+\rho_l\mathbf{u}_l\right)+\nabla\cdot\left(\sum\rho_{gi}\mathbf{u}_{gi}\mathbf{u}_{gi}+\rho_l\mathbf{u}_l\right)+\nabla p=\nabla\cdot\overleftrightarrow{\boldsymbol{\tau}}_v+\mathbf{m}_{eg}+\mathbf{m}_{el} \tag{A.4}$$

Global Conservation of Energy

$$\frac{\partial}{\partial t}\left(\sum\rho_{gi}e_{goi}+\rho_l e_{ol}\right)+\nabla\cdot\left(\sum\rho_{gi}\mathbf{u}_{gi}e_{goi}+\rho_l\mathbf{u}_l e_{ol}\right)+\nabla\cdot\left(\sum p_i\mathbf{u}_{gi}\right)=\nabla\cdot\left(\overleftrightarrow{\boldsymbol{\tau}}_v\cdot\mathbf{u}_g\right)-\nabla\cdot\mathbf{q}+Q_e \tag{A.5}$$

Equation of State, Gas Phase

$$p=\rho_g R_g T_g \tag{A.6}$$

For simplification, the above equations already contain some terms involving mass averaging over the species comprising the gas phase, namely the viscous tensor $\overleftrightarrow{\boldsymbol{\tau}}_v$; the vector $\mathbf{q}$ representing heat conduction; and the equation, of state (A.6). For more complete derivations of the equations for multicomponent mixtures, see for example Chapman and Cowling (1939); Hirschfelder, Curtis and Bird (1954); Toupin and Truesdell (1960); and Williams (1985). Superscipt $(\ )^{(l)}$ means that the liquid or condensed phase is the source and subscript $(\ )_e$ denotes an external source. Thus $w_g^{(l)}$ represents the rate at which gas is generated from liquid, and w_{ge} stands for the rate at which gas is generated by an external source. It follows from repeated use of the Gibbs-Dalton law for mixtures of perfect gases that p is the sum of partial pressures; ρ_g is the sum of the densities; and R is the mass average of the individual gas species, so for the gas phase we have

$$p=\sum p_i$$

$$\rho_g=\sum\rho_{gi} \tag{A.7) a,b,c}$$

$$R_g=\frac{1}{\rho_g}\sum\rho_{gi}R_i$$

Subscript $(\ )_i$ identifies the ith gaseous species; and in all cases except T_g, $(\ )_g$ means a mass average over all gas species as, for example,

$$\mathbf{u}_g=\frac{1}{\rho_g}\sum\rho_{gi}\mathbf{u}_{gi}=\sum Y_{gi}\mathbf{u}_{gi} \tag{A.8}$$

where $Y_{gi}=\rho_{gi}/\rho_g$ is the mass concentration of the ith species.

A.2. Mass, Momentum and Energy Transfer by Diffusion

The chief reasons for writing equations (A.1)–(A.5) explicitly with sums over species is to show the way in which diffusion of species arises; and to introduce the formula for the energy released by chemical reactions written in the conventional fashion. When definite problems are addressed, some approximations will be made to eliminate explicit dependence on diffusion velocities, but here we retain diffusion explicitly. Let $\mathbf{V}_i$ denote the diffusion velocity of the ith species with the definition

$$\mathbf{u}_{gi} = \mathbf{u}_g + \mathbf{V}_i \tag{A.9}$$

Multiply by ρ_{gi} and sum:

$$\sum \rho_{gi}\mathbf{u}_{gi} = \left(\sum \rho_{gi}\right)\mathbf{u}_g + \sum \rho_{gi}\mathbf{V}_i$$

Thus with (A.7)b and (A.8), this relation becomes

$$\sum \rho_{gi}\mathbf{V}_i = 0 \tag{A.10}$$

With $\rho_{gi} = \rho_g Y_{gi}$ and $\mathbf{u}_{gi} = \mathbf{u}_g + \mathbf{V}_i$, and substitution of equation (A.2) for global conservation of mass, the species equation (A.1) becomes

$$\rho_g \frac{\partial Y_{gi}}{\partial t} + \rho_g \mathbf{u}_g \cdot \nabla Y_{gi} = -\nabla \cdot (\rho_g \mathbf{V}_i Y_{gi}) + w_{gi} + w_{gi}^{(l)} + w_{gei} \tag{A.11}$$

In the momentum equation, $\sum \rho_{gi}\mathbf{u}_{gi} = \rho_g \mathbf{u}_g$ and with (A.10),

$$\sum \rho_{gi}\mathbf{u}_{gi}\mathbf{u}_{gi} = \sum \rho_{gi}\left(\mathbf{u}_g + \mathbf{V}_i\right)\left(\mathbf{u}_g + \mathbf{V}_i\right) = \sum \rho_{gi}\left(\mathbf{u}_g\mathbf{u}_g + \mathbf{V}_i\mathbf{u}_{gi} + \mathbf{u}_{gi}\mathbf{V}_i + \mathbf{V}_i\mathbf{V}_i\right)$$

Hence (A.4) is

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho_g\mathbf{u}_g + \rho_l\mathbf{u}_l\right) + \nabla \cdot \left(\rho_g\mathbf{u}_g + \rho_l\mathbf{u}_l\right) + \nabla p = \nabla \cdot \overleftrightarrow{\boldsymbol{\tau}}_v + \mathbf{m}_{eg} + \mathbf{m}_{el} \\ - \sum \left\{\nabla \cdot \left(\rho_{gi}\mathbf{V}_i\mathbf{u}_{gi} + \rho_{gi}\mathbf{u}_{gi}\mathbf{V}_i\right) + \nabla \cdot \left(\rho_{gi}\mathbf{V}_i\mathbf{V}_i\right)\right\}\end{aligned} \tag{A.12}$$

Diffusion in a multicomponent mixture generally involves coupling among all species, accounted for by introducing a matrix of diffusion coefficients and Fick's law relating the diffusion velocities to the gradients of species concentrations. That representation introduces unnecessary complications in the present context. We use the common approximation that the diffusion velocity of every species is proportional to the concentration gradient of that species only, and obeys Fick's law of diffusion in a binary mixture,

$$\rho_{gi}\mathbf{V}_i = -\rho_g \mathcal{D} \nabla Y_{gi} \tag{A.13}$$

where $\mathcal{D}$ is the binary diffusion coefficient assumed to have the same value for all species.

Even with the simple law (A.13), the momentum equation (A.12) is formally coupled nonlinearly to the species concentrations and to the total velocities $\mathbf{u}_{gi}$ of the species. Let $\mathbf{m}_D$ denote the terms involving diffusion,

$$\mathbf{m}_D = -\sum \{\nabla \cdot (\rho_{gi}\mathbf{V}_i\mathbf{u}_{gi} + \rho_{gi}\mathbf{u}_{gi}\mathbf{V}_i) + \nabla \cdot (\rho_{gi}\mathbf{V}_i\mathbf{V}_i)\} \tag{A.14}$$

and the global momentum equation is

$$\frac{\partial}{\partial t}(\rho_g\mathbf{u}_g + \rho_l\mathbf{u}_l) + \nabla \cdot (\rho_g\mathbf{u}_g\mathbf{u}_g + \rho_l\mathbf{u}_l\mathbf{u}_l) + \nabla p = \nabla \cdot \overleftrightarrow{\boldsymbol{\tau}}_v + \mathbf{m}_e + \mathbf{m}_D \tag{A.15}$$

where

$$\mathbf{m}_e = \mathbf{m}_{eg} + \mathbf{m}_{el} \tag{A.16}$$

is the total momentum addition from external sources.

The viscous stress tensor $\overleftrightarrow{\boldsymbol{\tau}}_v$ has a complicated form for a multicomponent mixture (Hirschfelder, Curtis and Bird, 1954). When necessary we will use a simple approximation. Unless shock waves are present, viscous effects internal to the flow field are negligible for combustion instabilities; they cannot be ignored at the boundaries.

Similar rearrangements of the energy equation (A.5) will lead to a form corresponding to (A.15). Substitute (A.8) in the first term of (A.5) and expand to give

$$\sum \rho_{gi}e_{goi} = \sum \rho_{gi}\left(e_{gi} + \frac{(\mathbf{u}_g + \mathbf{V}_i)^2}{2}\right) = \rho_g e_{og} + \sum \rho_{gi}\frac{\mathbf{V}_i^2}{2} \tag{A.17}$$

where (A.10) has been applied and the mass averaged stagnation energy is

$$\rho_g e_{og} = \sum \rho_{gi}e_{goi} \tag{A.18}$$

The term representing the total rate of work done by the partial pressures is

$$\sum p_i\mathbf{u}_{gi} = \sum p_i(\mathbf{u}_g + \mathbf{V}_i) = p\mathbf{u}_g + \sum p_i\mathbf{V}_i \tag{A.19}$$

and the net effect of convection of energy by the motions of the individual species can be written in the form:

$$
\begin{aligned}
\sum \rho_{gi}\mathbf{u_{gi}}e_{goi} &= \sum \rho_{gi}\mathbf{u}_g\left(e_{gi}+\frac{\mathbf{u}_{gi}^2}{2}\right)+\sum \rho_{gi}\mathbf{V}_i e_{goi} \\
&= \rho_g\mathbf{u}_g e_{og}+\mathbf{u}_g\sum\frac{1}{2}\rho_{gi}\left(\mathbf{u}_g+\mathbf{V}_i\right)^2+\sum\rho_{gi}\mathbf{V}_i e_{goi} \\
&= \rho_g\mathbf{u}_g e_{og}+\mathbf{u}_g\sum\rho_{gi}\mathbf{u}_g\cdot\mathbf{V}_i+\mathbf{u}_g\sum\frac{1}{2}\rho_{gi}\mathbf{V}_i^2+\sum\rho_{gi}\mathbf{V}_i e_{goi} \\
&= \rho_g\mathbf{u}_g e_{og}+\mathbf{u}_g\sum\frac{1}{2}\rho_{gi}\mathbf{V}_i^2+\sum\rho_{gi}\mathbf{V}_i e_{goi}
\end{aligned}
\tag{A.20}
$$

Substitution of (A.17), (A.19) and (A.20) in (A.5) leads to

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_g e_{og}\right)+\nabla\cdot\left(\rho_g\mathbf{u}_g e_{og}\right)+\frac{\partial}{\partial t}\left(\rho_l e_{ol}\right)+\nabla\cdot\left(\rho_l\mathbf{u}_l e_{ol}\right)+\nabla\cdot\left(p\mathbf{u}_g\right)+ \\
&+\left[\frac{\partial}{\partial t}\left(\sum\frac{1}{2}\rho_{gi}\mathbf{V}_i^2\right)+\nabla\cdot\left\{\mathbf{u}_g\sum\frac{1}{2}\rho_{gi}\mathbf{V}_i^2+\sum\left(\rho_{gi}\mathbf{V}_i e_{goi}+p_i\mathbf{V}_i\right)\right\}\right]= \\
&=\nabla\cdot\left(\overleftrightarrow{\boldsymbol{\tau}}_v\cdot\mathbf{u}_g\right)-\nabla\cdot\mathbf{q}+Q_e
\end{aligned}
\tag{A.21}
$$

Let χ_D denote all terms depending explicitly on the diffusion velocities;

$$
-\chi_D=\frac{\partial}{\partial t}\left(\sum\frac{1}{2}\rho_{gi}\mathbf{V}_i^2\right)+\nabla\cdot\left\{\mathbf{u}_g\sum\frac{1}{2}\rho_{gi}\mathbf{V}_i^2+\sum\left(\rho_{gi}\mathbf{V}_i e_{goi}+p_i\mathbf{V}_i\right)\right\}
\tag{A.22}
$$

and the energy equation is

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\rho_g e_{og}\right)+\nabla\cdot\left(\rho_g\mathbf{u}_g e_{og}\right)+\frac{\partial}{\partial t}\left(\rho_l e_{ol}\right)+\nabla\cdot\left(\rho_l\mathbf{u}_l e_{ol}\right)+\nabla\cdot\left(\rho_l\mathbf{u}_l e_{ol}\right)+\nabla\cdot\left(p\mathbf{u}_g\right)=\nabla\cdot\left(\overleftrightarrow{\boldsymbol{\tau}}_v\cdot\mathbf{u}_g\right)-\nabla\cdot\mathbf{q} \\
+Q_e+\chi_D
\end{aligned}
\tag{A.23}
$$

A.3. Construction of the Single Fluid Model

The discussion so far has been concerned with a medium consisting of two phases: the condensed phase denoted by subscript $(\quad)_l$ and the gas phase, identified by subscript $(\quad)_g$ comprising many species. For a broad range of conditions in practical propulsion systems, the volume fraction occupied by the condensed phase is small compared with that for the gas phase, although the mass fraction may be large. For example, if the propellant contains 19% aluminum, the combustion products in a solid rocket will contain about 38% by mass of condensed aluminum oxide. In liquid fueled combustors (gas turbines, afterburners and liquid rockets) limited regions may contain large volumetric fractions of liquid, but generally the volume occupied by the gaseous species is the greater part.

Under these conditions it is a valid approximation to use a 'single-fluid model' of the medium, a representation in which the density is the sum of the densities (mass per unit volume) of gas and condensed phase, $\rho = \rho_g + \rho_l$; the compressibility and pressure are those of the gas phase only; and the thermodynamic properties are mass averages of the gas and liquid phases. The chief immediate consequence is that the speed of sound for the single fluid correctly takes into account the primary individual effects of the gas and liquid phases: The sound speed is reduced by the presence of condensed material. Furthermore, the flow of the medium is dominated by the gas phase, with perturbations of its motion arising from interactions between the gas and condensed phases. We therefore seek a system of equations in which the density is the sum

$$\rho = \rho_g + \rho_l \ . \tag{A.24}$$

Eventually the primary dependent variables for investigating the dynamics of a combustor will be ρ and the pressure, density and temperature of the gas.[3]

A.3.1. Equation for the Density. To find an equation for the density, add (A.2) and (A.3):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_g \mathbf{u}_g + \rho_l \mathbf{u}_l) = w_e \tag{A.25}$$

where $\rho = \rho_g + \rho_l$ and the external source of mass is

$$w_e = w_{eg} + w_{el} \tag{A.26}$$

Add $\nabla \cdot (\rho_l \mathbf{u}_g)$ to both sides of (A.25) and rearrange to give the desired equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = \mathcal{W} \tag{A.27}$$

where we have set $\mathbf{u} := \mathbf{u}_g$ and the source is

$$\mathcal{W} = w_e - \nabla \cdot (\rho_l \delta \mathbf{u}) \tag{A.28}$$

The 'slip velocity' is

$$\delta \mathbf{u}_l = \mathbf{u}_l - \mathbf{u}_g \tag{A.29}$$

[3]The basic idea of constructing a single-phase model seems to have originated in early work in two-phase flow, e.g. Rannie (1962) and Marble (1963, 1970)

A.3.2. Equation for Momentum. The momentum equation for the single-fluid model may be found by expanding the left-hand side of (A.15) and substituting equations (A.2) and (A.3); rearrangement gives

$$\rho_g \left[\frac{\partial \mathbf{u}_g}{\partial t} + \mathbf{u}_g \cdot \nabla \mathbf{u}_g \right] + \nabla p = \nabla \cdot \overleftrightarrow{\boldsymbol{\tau}}_v + \mathbf{m}_e + \mathbf{m}_D - \boldsymbol{\sigma}_e + \delta \mathbf{u}_l w_g^{(l)} + \mathbf{F}_l \tag{A.30}$$

where

$$\boldsymbol{\sigma}_e = \mathbf{u}_g w_{eg} + \mathbf{u}_l w_{el} \tag{A.31}$$

$$\mathbf{F}_l = -\rho_l \left[\frac{\partial \mathbf{u}_l}{\partial t} + \mathbf{u}_l \cdot \nabla \mathbf{u}_l \right] \tag{A.32}$$

Now add $\rho_l \left[\frac{\partial \mathbf{u}_g}{\partial t} + \mathbf{u}_g \cdot \nabla \mathbf{u}_g \right]$ to both sides of (A.30) and combine terms on the right-hand side to give

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] + \nabla p = \boldsymbol{\mathcal{F}} \tag{A.33}$$

where

$$\boldsymbol{\mathcal{F}} = \nabla \cdot \overleftrightarrow{\boldsymbol{\tau}}_v + \mathbf{m}_e + \mathbf{m}_D - \boldsymbol{\sigma}_e - \delta \mathbf{u}_l w_g^{(l)} - \rho_l \frac{D \delta u_l}{Dt} \tag{A.34}$$

$$\mathbf{u} := \mathbf{u}_g \tag{A.35}$$

and $\mathbf{u}$ has been written for $\mathbf{u}_g$.

A.3.3. Equation for Energy. The energy equation must be handled somewhat differently in order to introduce the conventional definition of the heat of reaction and the mass-averaged thermodynamic properties. Denbigh (1961) provides good accessible discussions of these matters; it's a good reference for basic chemical thermodynamics. First replace the energy e_{og} by the enthalpy $h_{og} = e_{og} - p/\rho_g$ in (A.21) and use equation (A.2) to find

$$\rho_g \left[\frac{\partial h_{og}}{\partial t} + \mathbf{u}_g \cdot \nabla h_{og} \right] - \rho_g \left[\frac{\partial}{\partial t} \left(\frac{p}{\rho_g} \right) + \mathbf{u}_g \cdot \nabla \left(\frac{p}{\rho_g} \right) + \nabla \cdot (p \mathbf{u}_g) \right] = H \tag{A.36}$$

where H stands temporarily for the remaining terms,

$$H = \nabla \cdot \left(\overleftrightarrow{\boldsymbol{\tau}}_v \cdot \mathbf{u}_g \right) - \nabla \cdot \mathbf{q} - (e_{og} w_g + e_{ol} w_l) - \rho_l \left[\frac{\partial e_{ol}}{\partial t} + \mathbf{u}_l \cdot \nabla e_{ol} \right] + Q_e + \chi_D$$

The enthalpy of the mixture is

$$h_g = \frac{1}{\rho_g} \sum \rho_{gi} h_i = \sum Y_{gi} h_i \tag{A.37}$$

with

$$h_i = h_i^{(r)} + \int_{T_r}^{T} c_{\rho i} dT' = h_i^{(r)} + h_i^{(T)} \tag{A.38}$$

and $h_i^{(r)}$ is the enthalpy of the ith species at the reference temperature T_r; $h_i^{(T)}$ is sometimes called the 'thermal enthalpy'. Because the $h_i^{(r)}$ are constants here,

$$\begin{aligned} \frac{\partial h_{og}}{\partial t} + \mathbf{u}_g \cdot \nabla h_{og} &= \left[\frac{\partial h_g}{\partial t} + \mathbf{u}_g \cdot \nabla h_g \right] + \left[\frac{\partial}{\partial t} \left(\frac{u_g^2}{2} \right) + \mathbf{u}_g \cdot \nabla \left(\frac{u_g^2}{2} \right) \right] \\ &= \sum Y_{gi} \left[\frac{\partial h_i^{(T)}}{\partial t} + \mathbf{u}_g \cdot \nabla h_i^{(T)} \right] + \left[\frac{\partial}{\partial t} \left(\frac{u_g^2}{2} \right) + \mathbf{u}_g \cdot \nabla \left(\frac{u_g^2}{2} \right) \right] \\ &\quad + \frac{1}{\rho_g} \sum \left(h_i^{(r)} + h_i^{(T)} \right) \left[-\nabla \cdot (\rho_g \mathbf{V}_i Y_{gi}) + w_{gi} + w_{gi}^{(l)} + w_{gei} \right] \end{aligned} \tag{A.39}$$

after (A.11) has been substituted in the last summation. The sum $\sum \left(h_i^{(r)} + h_i^{(T)} \right) w_{gi}$ is related to the heat of reaction at the local temperature and pressure. The precise connection is not important here and for simplicity we will define the heat of reaction Q and reaction rate w by

$$-Qw = \sum \left(h_i^{(r)} + h_i^{(T)} \right) \left(w_{gi} + w_{gi}^{(l)} + w_{gei} \right) \tag{A.40}$$

Substitution in (A.37) and defining

$$h_0^{(T)} = h^{(T)} + \frac{\mathbf{u}_g^2}{2} \tag{A.41}$$

we have

$$\rho_g \left[\frac{\partial h_{og}}{\partial t} + \mathbf{u}_g \cdot \nabla h_{og} \right] = \rho_g \left[\frac{\partial h_{og}^{(T)}}{\partial t} + \mathbf{u}_g \cdot \nabla h_{og}^{(T)} \right] - Qw - \sum h_i \nabla \cdot (\rho_g \mathbf{V}_i Y_{gi}) \tag{A.42}$$

Consistent with the definition (A.38), the thermal internal energy is $e^{(T)} = h^{(T)} + p/\rho_g$ and

$$e_{og}^{(T)} = e^{(T)} + \frac{1}{2}\mathbf{u}_g^2 \tag{A.43}$$

Substitution of (A.42) and (A.43) in (A.36) and use of (A.3) leads to the new form of the energy equation

$$\begin{aligned}\rho_g\left[\frac{\partial e_{og}^{(T)}}{\partial t} + \mathbf{u}_g\cdot\nabla e_{og}^{(T)}\right] + \rho_l\left[\frac{\partial e_l}{\partial t} + \mathbf{u}_g\cdot\nabla e_l\right] + \nabla\cdot(p\mathbf{u}_g) = \nabla\cdot\left(\overleftrightarrow{\boldsymbol{\tau}}_v\cdot\mathbf{u}_g\right) - \nabla\cdot\mathbf{q} + Qw + Q_e \\ - (e_{og}w_g + e_{ol}w_l) + \chi_D + \sum h_i\nabla\cdot(\rho_g\mathbf{V}_iY_{gi})\end{aligned} \tag{A.44}$$

where $w_l = w_{el} - w_g^{(l)}$.

Now remove the kinetic energies of the gas and liquid phases from this equation by subtracting the scalar products of $\mathbf{u}_g$ with (A.31) and $\mathbf{u}_l$ with (A.33). to give

$$\begin{aligned}\rho_gC_v\left[\frac{\partial T_g}{\partial t} + \mathbf{u}_g\cdot\nabla T_g\right] + \rho_lC_l\left[\frac{\partial T_l}{\partial t} + \mathbf{u}_l\cdot\nabla T_l\right] + p\nabla\cdot\mathbf{u}_g = \\ = \nabla\cdot\left(\overleftrightarrow{\boldsymbol{\tau}}_v\cdot\mathbf{u}_g\right) - \nabla\cdot\mathbf{q} + Qw + Q_e - (e_{og}w_g + e_{ol}w_l) + \chi_D + \sum h_i\nabla\cdot(\rho_g\mathbf{V}_iY_{gi}) \\ -\mathbf{u}\cdot\boldsymbol{\mathfrak{F}} + \mathbf{u}_l\cdot\mathbf{F}_l\end{aligned} \tag{A.45}$$

where the specific heats are defined by

$$\begin{aligned}de_g^{(T)} &= C_v dT_g \\ de_l &= C_l dT_l\end{aligned} \tag{A.46 a,b}$$

Rearrangement of (A.45) in a manner similar to that used for rewriting the momentum equation in the form (A.34) produces the result

$$\rho\mathbf{C}_v\left[\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T\right] + p\nabla\cdot\mathbf{u} = \mathcal{Q} \tag{A.47}$$

where the symbol T_g has been replaced by T[4] and

$$\begin{aligned}\mathcal{Q} = \overleftrightarrow{\boldsymbol{\tau}}_v\cdot\nabla\cdot\mathbf{u} - \nabla\cdot\mathbf{q} + Qw + Q_e - (e_{og}w_g + e_{ol}w_l) + \chi_D + \sum h_i\nabla\cdot(\rho_g\mathbf{V}_iY_{gi}) \\ - \mathbf{u}\cdot(\mathbf{m}_e + \mathbf{m}_D - \boldsymbol{\sigma}_e) + (\mathbf{u}\cdot\delta\mathbf{u})\,w_g^{(l)} + \delta Q_l + \delta\mathbf{u}\cdot\mathbf{F}_l - \mathbf{u}\cdot(\boldsymbol{\mathfrak{F}} - \mathbf{F}_l)\end{aligned} \tag{A.48}$$

$$\delta Q_l = -\rho_l\mathbf{C}_l\left[\frac{\partial\delta T}{\partial t} + \mathbf{u}_l\cdot\nabla\delta T + \delta\mathbf{u}_l\cdot\nabla T\right] \tag{A.49}$$

with the definitions

$$\delta T = T_l - T \quad ; \quad \mathbf{m} = \mathbf{m}_e + \mathbf{m}_D \tag{A.50}$$

$$\mathbf{C}_v = \frac{\rho_gC_v + \rho_lC_l}{\rho_g + \rho_l} = \frac{C_v + C_mC_l}{1 + C_m} \tag{A.51}$$

The mass fraction $C_m = \rho_l/\rho_g$ is often called the *particle loading* because it has commonly appeared in problems involving flows of gases containing small solid or liquid particles.

[4]Thus, in contrast with the density (equation A.25), the temperatures of the gaseous and condensed phases are not formally combined and the temperature of the condensed phase must be calculated separately.

A.3.4. Equation for the Pressure. Let R_g denote the gas constant for the mixture and add T times (A.27) to $\mathbf{C}_v^{-1}$ times (A.47) to find

$$\frac{\partial}{\partial t}(\rho T) + \mathbf{u}\cdot\nabla(\rho T) + \left(\rho T + \frac{p}{\mathbf{C}_v}\right)\nabla\cdot\mathbf{u} = T\mathcal{W} - T\nabla\cdot(\rho_l\delta\mathbf{u}) + \frac{1}{\mathbf{C}_v}\mathcal{Q} \tag{A.52}$$

The two-phase mixture behaves as a perfect gas having density $\rho_g + \rho_l$ and mass averaged properties if we assume that the mixture of gases only obeys the Gibbs-Dalton law, (A.6):

$$p = R_g\rho_g T \tag{A.53}$$

where R_g is the mass-averaged gas constant. Now multiply and divide the right side by $\rho_g + \rho_l = \rho$ and set $T_g = T$ to give

$$p = \mathbf{R}\rho T \tag{A.54}$$

where

$$\mathbf{R} = \frac{R_g}{1 + C_m} \tag{A.55}$$

With (A.51), the last implies that the specific heat at constant volume and $\boldsymbol{\gamma} = \mathbf{C}_p/\mathbf{C}_v$ for the two-phase mixture are:

$$\mathbf{C}_v = \frac{C_v + C_m C_l}{1 + C_m}$$

$$\boldsymbol{\gamma} = \frac{C_p + C_m C_l}{C_v + C_m C_l} \tag{A.56) a,b}$$

For several reasons clarified in the main text, the theory of combustion instabilities discussed in this book is based on the wave equation for the pressure derived in Chapter 3. The first order equation for the pressure is derived by adding T times equation (A.27) to $\mathbf{C}_v^{-1}$ times (A.48):

$$\frac{\partial}{\partial t}(\rho T) + \mathbf{u}\cdot\nabla(\rho T) + \left(\frac{p}{\mathbf{C}_v} + \rho T\right)\nabla\cdot\mathbf{u} = T\left[\mathcal{W} - \nabla\cdot(\rho_l\delta\mathbf{u}_l)\right] + \frac{1}{\mathbf{C}_v}\mathcal{Q}$$

Hence with (A.55) we find

$$\frac{\partial p}{\partial t} + \boldsymbol{\gamma} p\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla p = \mathcal{P} \tag{A.57}$$

where

$$\mathcal{P} = \frac{\mathbf{R}}{\mathbf{C}_v}\mathcal{Q} + \mathbf{R}T\left[\mathcal{W} - \nabla\cdot(\rho_l\delta\mathbf{u}_l)\right] \tag{A.58}$$

A.4. Muster of Equations

The set of equations forming the basis for all of the theory and analysis we discuss in this book is:

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u} + \mathcal{W} \tag{A.59}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \boldsymbol{\mathcal{F}} \tag{A.60}$$

$$\rho \mathbf{C}_v \frac{DT}{Dt} = -p\nabla \cdot \mathbf{u} + \mathcal{Q} \tag{A.61}$$

$$\frac{Dp}{Dt} = -\boldsymbol{\gamma} p \nabla \cdot \mathbf{u} + \mathcal{P} \tag{A.62}$$

$$\frac{Ds}{Dt} = \frac{1}{T}\mathcal{S} \tag{A.63}$$

$$p = \mathbf{R}\rho T \tag{A.64}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$ is the connective operator based on the mass-averaged velocity u of the gases.

For completeness we have also included the equation (A.63) for the entropy, obtained in familiar fashion by applying the combined First and Second Laws of Thermodynamics to an element of fluid. That is, the relation $de = Tds - pdv$ can be written

$$\begin{aligned} \frac{Ds}{Dt} &= \frac{1}{T}\left(\frac{De}{Dt} + p\frac{Dv}{Dt}\right) \\ &= \frac{1}{T}\left(\mathbf{C}_v \frac{DT}{Dt} + p\frac{Dv}{Dt}\right) \end{aligned} \tag{A.65}$$

Substitution of (A.59) and (A.61) gives (A.63) with the source

$$\mathcal{S} = \mathcal{Q} - \frac{p}{\rho^2}\mathcal{W} \tag{A.66}$$

It is important to realize that this formulation contains all relevant physical processes, including those representing the actions of external influences associated, for example, with active control of combustor dynamics.

The source functions in (A.59)–(A.63) are

$$\mathcal{W} = w_e - \nabla \cdot (\rho_l \delta \mathbf{u}_l) \tag{A.67}$$

$$\boldsymbol{\mathcal{F}} = \nabla \cdot \overleftrightarrow{\boldsymbol{\tau}}_v + \mathbf{m}_e + \mathbf{m}_D - \boldsymbol{\sigma}_e - \delta \mathbf{u}_l w_g^{(l)} - \rho_l \frac{D\delta u_l}{Dt} \tag{A.68}$$

$$\begin{aligned} \mathcal{Q} = {} & \overleftrightarrow{\boldsymbol{\tau}}_v \cdot \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q} + Qw + Q_e - (e_{og} w_g + e_{ol} w_l) + \chi_D + \sum h_i \nabla \cdot (\rho_g \mathbf{V}_i Y_{gi}) \\ & - \mathbf{u} \cdot (\mathbf{m}_e + \mathbf{m}_D - \boldsymbol{\sigma}_e) + (\mathbf{u} \cdot \delta \mathbf{u}_l)\, w_g^{(l)} + \delta Q_l + \delta \mathbf{u}_l \cdot \mathbf{F}_l - \mathbf{u} \cdot (\boldsymbol{\mathcal{F}} - \mathbf{F}_l) \end{aligned} \tag{A.69}$$

$$\mathcal{P} = \frac{\mathbf{R}}{\mathbf{C}_v}\mathcal{Q} + \mathbf{R}T\left[\mathcal{W} - \nabla\cdot(\rho_l\delta\mathbf{u}_l)\right] \tag{A.70}$$

$$\mathcal{S} = \mathcal{Q} - \frac{p}{\rho^2}\mathcal{W} \tag{A.71}$$

Recall that Q and w are defined together as the heat (and enthalpy) supplied per unit volume by chemical reactions, equation (A.40). Also, we have set $u_g := u$ and $T_g := T$.

A.5. Expansions in Mean and Fluctuating Variables

Following the steps suggested in Section 3.3 to produce equations (3.22)–(3.27) will give the expressions for the brackets defined there to simplify the appearance of the equations. As an intermediate step it is helpful to introduce the definitions:

$$\{\rho\}^1 = \frac{\bar{D}\rho'}{Dt} + \bar{\rho}\nabla\cdot\mathbf{M}' + \rho'\nabla\cdot\bar{\mathbf{M}} + \mathbf{M}'\cdot\nabla\bar{\rho} - \boldsymbol{\mathcal{W}}'_1 \tag{A.72}$$

$$\{\rho\}^2 = \mathbf{M}'\cdot\nabla\rho' + \rho'\nabla\cdot\mathbf{M}' - \boldsymbol{\mathcal{W}}'_2 \tag{A.73}$$

$$\{M\}^1 = \bar{\rho}\frac{\bar{D}\bar{\mathbf{M}}}{Dt} + \bar{\rho}\mathbf{M}'\cdot\nabla\mathbf{M}' + \nabla p' + \rho'\frac{\bar{D}\bar{\mathbf{M}}}{Dt} - \boldsymbol{\mathcal{F}}'_1 \tag{A.74}$$

$$\{M\}^2 = \rho'\frac{\bar{D}\mathbf{M}'}{Dt} + \rho'\mathbf{M}'\cdot\nabla\bar{\mathbf{M}} + \bar{\rho}\mathbf{M}'\cdot\nabla\mathbf{M}' - \boldsymbol{\mathcal{F}}'_2 \tag{A.75}$$

$$\{M\}^3 = \rho'\mathbf{M}'\cdot\nabla\mathbf{M}' - \boldsymbol{\mathcal{F}}'_3 \tag{A.76}$$

$$\{T\}^1 = \bar{\rho}\mathbf{C_v}\frac{\bar{D}T'}{Dt} + \bar{\rho}\mathbf{C_v}\mathbf{M}'\cdot\nabla\mathbf{M}' + \nabla\bar{T} + \bar{p}\nabla\mathbf{M}' + \rho'\mathbf{C_v}\frac{\bar{D}\bar{T}}{Dt} - \mathcal{Q}'_1 \tag{A.77}$$

$$\{T\}^2 = \rho'\mathbf{C_v}\frac{\bar{D}T'}{Dt} + \rho'\mathbf{C_v}\mathbf{M}'\cdot\nabla\bar{T} + \bar{\rho}\mathbf{C_v}\mathbf{M}'\cdot\nabla T' - \mathcal{Q}'_2 \tag{A.78}$$

$$\{T\}^3 = \rho'\mathbf{C_v}\mathbf{M}'\cdot\nabla T' - \mathcal{Q}'_3 \tag{A.79}$$

$$\{p\}^1 = \frac{\bar{D}p'}{Dt} + \gamma\bar{\rho}\nabla\cdot\mathbf{M}' + \gamma p'\nabla\cdot\bar{\mathbf{M}} - \mathcal{P}'_1 \tag{A.80}$$

$$\{p\}^2 = \mathbf{M}'\cdot\nabla p' + \gamma p'\nabla\cdot\mathbf{M}' - \mathcal{P}'_2 \tag{A.81}$$

$$\{s\}^1 = \bar{\rho}\bar{T}\frac{\bar{D}s'}{Dt} + \left(\rho'\bar{T} + \bar{\rho}T'\right)\frac{\bar{D}\bar{s}}{Dt} + \bar{\rho}\bar{T}\mathbf{M}'\cdot\nabla\bar{s} - \mathcal{S}'_1 \tag{A.82}$$

$$\{s\}^2 = \left(\rho'\bar{T} + \bar{\rho}T'\right)\frac{\bar{D}\bar{s}}{Dt} + \bar{\rho}\bar{T}\mathbf{M}'\cdot\nabla s' + \rho'T'\bar{\mathbf{M}}\cdot\nabla s' + \left(\rho'\bar{T} + \bar{\rho}T'\right)\mathbf{M}'\cdot\nabla\bar{s} - \mathcal{S}_2' \tag{A.83}$$

$$\{s\}^3 = \rho'T'\frac{\bar{D}s'}{Dt} + \rho'T'\bar{\mathbf{M}}\cdot\nabla s' + \rho'T'\mathbf{M}'\cdot\nabla\bar{s} - \mathcal{S}_3' \tag{A.84}$$

$$\{s\}^4 = \rho'T'\mathbf{M}'\cdot\nabla s' - \mathcal{S}_4' \tag{A.85}$$

where $\overline{D}/Dt = \partial/\partial t + \overline{M}\cdot\nabla$ as defined in Chapter 3, equation (3.21), is the dimensionless convective operator based on the steady mass-averaged velocity. Superscript $\{\quad\}^n$ and subscript $(\quad)_n$ always mean here that the indicated quantities are to be written to order n in the fluctuations of the flow variables.

The fluctuations of the source functions $\mathcal{W}, \cdots, \mathcal{S}$ will be formed as required for particular applications and cannot be written until literal models are constructed.

A.6. Definitions of the Ordering Brackets

The brackets introduced in equations (3.22)–(3.27) are obtained by rearrangement of the sums of brackets defined in Section A.5:

$$\{\rho\}^1 + \{\rho\}^2 = \{[\rho]\}_1 + \{\rho\}_2 - \boldsymbol{\mathcal{W}}' \tag{A.86}$$

$$\{\mathbf{M}\}^1 + \{\mathbf{M}\}^2 + \{\mathbf{M}\}^3 = \{[\mathbf{M}]\}_1 + \{\mathbf{M}\}_2 + \{[\mathbf{M}]\}_2 + \{\mathbf{M}\}_3 - \boldsymbol{\mathcal{F}}' \tag{A.87}$$

$$\{T\}^1 + \{T\}^2 + \{T\}^3 = \{[T]\}_1 + \{T\}_2 + \{[T]\}_2 + \{T\}_3 - \mathcal{Q}' \tag{A.88}$$

$$\{p\}^1 + \{p\}^2 = \{[p]\}_1 + \{p\}_2 - \mathcal{P}' \tag{A.89}$$

$$\{s\}^1 + \{s\}^2 + \{s\}^3 + \{s\}^4 = \{[s]\}_1 + \{s\}_2 + \{[s]\}_2 + \{s\}_3 + \{s\}_4 - \mathcal{S}' \tag{A.90}$$

By identifying corresponding terms in these relations, we find[5]

[5] As in Chapter 3, bold symbols $\mathbf{R}$, $\mathbf{C}_v$ and $\boldsymbol{\gamma}$ are replaced here by the usual forms R, C_v, γ, but the symbols continue to represent mass-averaged properties.

$$\{[\rho]\}_1 = \bar{\mathbf{M}} \cdot \nabla \rho' + \mathbf{M}' \cdot \nabla \bar{\rho} + \rho' \nabla \cdot \bar{\mathbf{M}}$$

$$\{\rho\}_2 = \mathbf{M}' \cdot \nabla \rho' + \rho' \nabla \cdot \mathbf{M}'$$

$$\{[M]\}_1 = \bar{\rho} \left(\bar{\mathbf{M}} \cdot \nabla \mathbf{M}' + \mathbf{M}' \cdot \nabla \bar{\mathbf{M}} \right) + \rho' \frac{\bar{D}\bar{\mathbf{M}}}{Dt}$$

$$\{M\}_2 = \rho' \frac{\partial \mathbf{M}'}{\partial t} + \rho' (\bar{\mathbf{M}} \cdot \nabla \mathbf{M}' + \mathbf{M}' \cdot \nabla \bar{\mathbf{M}}) + \bar{\rho} \mathbf{M}' \cdot \nabla \mathbf{M}'$$

$$\{[M]\}_2 = \rho' \left(\bar{\mathbf{M}} \cdot \nabla \mathbf{M}' + \mathbf{M}' \cdot \nabla \bar{\mathbf{M}} \right)$$

$$\{M\}_3 = \rho' \mathbf{M}' \cdot \nabla \mathbf{M}'$$

$$\{[p]\}_1 = \bar{\mathbf{M}} \cdot \nabla p' + \gamma p' \nabla \cdot \bar{\mathbf{M}} + \mathbf{M}' \cdot \nabla \bar{p}$$

$$\{p\}_2 = \mathbf{M}' \cdot \nabla p' + \gamma p' \nabla \cdot \mathbf{M}'$$

$$\{[T]\}_1 = \bar{\rho} C_v \left(\bar{\mathbf{M}} \cdot \nabla T' + \mathbf{M}' \cdot \nabla \bar{T} \right) + \\ + C_v \rho' \frac{\bar{D}\bar{T}}{Dt} + p' \nabla \cdot \bar{\mathbf{M}}$$

$$\{T\}_2 = \rho' C_v \frac{\partial T'}{\partial t} + \bar{\rho} C_v \mathbf{M}' \cdot \nabla T' + p' \nabla \cdot \mathbf{M}'$$

$$\{[T]\}_2 = \rho' C_v \left(\bar{\mathbf{M}} \cdot \nabla T' + \mathbf{M}' \cdot \nabla \bar{T} \right)$$

$$\{T\}_3 = \rho' C_v \mathbf{M}' \cdot \nabla T'$$

$$\{[s]\}_1 = \bar{\rho} \bar{T} \left(\bar{\mathbf{M}} \cdot \nabla s' + \mathbf{M}' \cdot \nabla \bar{s} \right)$$

$$\{s\}_2 = \left(\bar{\rho} T' + \rho' \bar{T} \right) \frac{\partial s'}{\partial t} + \bar{\rho} \bar{T} \mathbf{M}' \cdot \nabla s' + \left(\rho' \bar{T} + \bar{\rho} T' \right) + (\mathbf{M}' \cdot \nabla \bar{s}) + \rho' T' \frac{\partial \bar{s}}{\partial t}$$

$$\{[s]\}_2 = \left(\bar{\rho} T' + \rho' \bar{T} \right) \bar{\mathbf{M}} \cdot \nabla s'$$

$$\{s\}_3 = \rho' T' \frac{\partial s'}{\partial t} + \left(\bar{\rho} T' + \rho' \bar{T} \right) \mathbf{M}' \cdot \nabla s' + \rho' T' \left(\mathbf{M}' \cdot \nabla \bar{s} \right) + \bar{\mathbf{M}} \cdot \nabla s'$$

$$\{s\}_4 = \rho' T' \mathbf{M}' \cdot \nabla s'$$

$$\{p - R\rho T\}_1 = p' - R(\rho' \bar{T} + \bar{\rho} T')$$

$$\{p - R\rho T\}_2 = -R\rho' T'$$

The subscript $\{\quad\}_n$ on the curly brackets means that the contained quantities are written to order n in the fluctuations of flow variables. Similarly, the square brackets indicate that the terms are of first order in the Mach number of the mean flow. Higher order square brackets are not required, as explained in Section 3.3.1.