ANNEX B

The Equations for One-Dimensional Unsteady Motions

These are many problems for which the flow may be approximated as one-dimensional. Even when the approximation may not seem as accurate as we might like, it is always a good beginning. The desired results are usually obtained without real effect and often are inspiringly close to the truth. An elementary example is computation of the normal modes for a straight tube having discontinuities, Section 5.7.2. Here we are concerned with situations in which influences at the lateral boundary must be accounted for. The formulation of the general problem is then essentially the counterpart of the constitution of the one-dimensional equations for steady flow in ducts thoroughly discussed by Shapiro (1950) and Crocco (1958).

Accounting for changes of area in the one-dimensional approximation is a straightforward matter; following the rules applied to derivations appearing in the three-dimensional equations:

$$\begin{aligned} \mathbf{u}\cdot\nabla(\quad) &\rightarrow u\frac{\partial}{\partial x}(\quad) \\ \nabla\cdot(\quad) &\rightarrow \frac{1}{S_c}\frac{\partial}{\partial x}S_c(\quad) \\ \nabla^2(\quad) &\rightarrow \frac{1}{S_c}\frac{\partial}{\partial x}S_c\frac{\partial(\quad)}{\partial x} \end{aligned} \tag{B.1}$$

where the axis of the duct lies along the x-direction and $S_c(x)$ is the distribution of the cross-section area.

More interesting are consequences of processes at the lateral boundary, particularly when there is flow through the surface. The most important applications arise in solid propellant rockets when burning propellant forms all or part of the lateral surface. Inflow of mass momentum and energy must be accounted for (Culick 1971, 1973; Culick and Yang 1990). The equations have the same form as the three-dimensional equations derived in Annex A, equations (A.59)–(A.63) but the rule (B.1) has been applied and only the velocity component u along axis of the duct is non-zero:

Conservation of Mass

$$\frac{D\rho}{Dt} = -\rho\frac{1}{S_c}\frac{\partial}{\partial x}(S_c u) + (\mathcal{W}_1 + \mathcal{W}_{1s}) \tag{B.2}$$

Conservation of Momentum

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + (\mathcal{F}_1 + \mathcal{F}_{1s}) \tag{B.3}$$

Conservation of Energy

$$\rho C_v\frac{DT}{Dt} = -p\frac{1}{S_c}\frac{\partial}{\partial x}(S_c u) + (\mathcal{Q}_1 + \mathcal{Q}_{1s}) \tag{B.4}$$

Equation for the Pressure

$$\frac{Dp}{Dt} = -\gamma p \frac{1}{S_c}\frac{\partial}{\partial x}(S_c u) + (\mathcal{P}_1 + \mathcal{P}_{1s}) \tag{B.5}$$

Equation for the Entropy

$$\frac{Ds}{Dt} = \frac{1}{T}(\mathcal{S}_1 + \mathcal{S}_{1s}) \tag{B.6}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{B.7}$$

The source terms $\mathcal{W}_1$, $\mathcal{F}_1$, $\mathcal{Q}_1$, $\mathcal{P}_1$ and $\mathcal{S}_1$ are the one-dimensional forms of (A.67)–(A.71) written for the axial component of velocity only and with the rules (B.1) applied:

$$\mathcal{W}_1 = -\frac{1}{S_c}\frac{\partial}{\partial x}(S_c \rho_l \delta u_l) + w_e \tag{B.8}$$

$$\mathcal{F}_1 = \frac{\partial \tau_v}{\partial x} + m_e + m_D - \sigma_e - \delta u_l w_g^{(l)} - \rho_l \frac{D\delta u_l}{Dt} \tag{B.9}$$

$$\begin{aligned}\mathcal{Q}_1 = \tau_v \frac{\partial u}{\partial x} - \frac{\partial q}{\partial x} + Qw + Q_e - (e_{og} w_g + e_{ol} w_l) \\ + \chi_0 + \Sigma h_i \frac{\partial}{\partial x}(\rho_g V_i Y_{gi}) - u(m_e + m_D - \sigma_e) \\ + (u\delta u_l) w_g^{(l)} + \delta Q_l + \delta u_l F_l - u(\mathcal{F}_1 - F_l)\end{aligned} \tag{B.10}$$

$$\mathcal{P}_1 = \frac{R}{C_v}\mathcal{Q}_1 + RT\left[\mathcal{W}_1 - \frac{\partial}{\partial x}(\rho_l \delta u_l)\right] \tag{B.11}$$

$$\mathcal{S}_1 = \mathcal{Q}_1 - \frac{p}{\rho^2}\mathcal{W}_1 \tag{B.12}$$

and F_l is the one-dimensional form of (A.32).

In addition, sources of mass, momentum and energy associated with flow through the lateral boundary are represented by the symbols with subscript $(\quad)_s$ (Culick 1973; Culick and Yang 1990):

$$\mathcal{W}_{1s} = \frac{1}{S_c}\int m_s dq = \frac{1}{S_c}\int [m_{sg} + m_{sl}]\, dq \tag{B.13}$$

$$\mathcal{F}_{1s} = \frac{1}{S_c}\int [(u_s - u)m_{sg} + (u_{ls} - u_l)m_{sl}]\, dq \tag{B.14}$$

$$\mathcal{Q}_{1s} = \frac{1}{S_c}\int [(h_{0s} - e_0)m_{sg} + (e_{l0s} - e_{l0})m_{sl}]\, dq \tag{B.15}$$

$$\mathcal{P}_{1s} = \frac{R}{C_v}\mathcal{Q}_{1s} + RT_s\mathcal{W}_{1s} = \frac{R}{C_v}\frac{1}{S_c}\int\left[(h_{0s}-e_0)m_{sg} + (e_{l0s}-e_{l0})m_{sl} + C_vTm_{sg}\right]dq \tag{B.16}$$

$$\mathcal{S}_{1s} = \frac{1}{\rho}\mathcal{Q}_1 - \frac{p}{\rho}\mathcal{W}_{1s} \tag{B.17}$$

A useful form for $\mathcal{P}_{1s}$ is

$$\mathcal{P}_{1s} = \frac{1}{S_c}\int(a^2+\gamma R\Delta T_s)m_{sg}dq + \frac{R}{C_v}\frac{1}{S_c}\int(e_{l0s}-e_{l0})m_{sl}\,dq \tag{B.18}$$

where $\Delta T_s = T_s - T$. Terms containing $u_s^2 m_{sg}$ or $u^2 m_{sg}$ are of higher order and therefore are negligible.

Superscripts $(\quad)_g$ and $(\quad)_l$ refer respectively to the gas and liquid phases and subscript $(\quad)_s$ denotes values at the surface. The mass fluxes at the surface, m_{sg} and m_{sl} are of course computed as values normal to the boundary and are positive for inward flow. Here q stands for the perimeter of the local section normal to the axis.

B.1. Equations for Unsteady One-Dimensional Motions

Forming the equations for the fluctuating motions within the one-dimensional approximation is done in exactly the same way as for the general equations, Annex A. We need only apply the rules (B.1) and add to the inhomogeneous functions h and f the contributions from processes at the boundary. As for the general three-dimensional equations, we defer writing the fluctuations $\mathcal{W}_1'$, $\mathcal{F}_1'$, ... until we consider specific problems.

The procedure introduced in Section 3.3.3 for forming the systems of equations for a hierarchy of problems applies equally to one-dimensional motions. As above, the equations are obtained from the three-dimensional equations by applying the rules (B.1): the results can be constructed when needed. However, the contributions from processes at the lateral boundary are special. Written to first order in the fluctuations and the Mach number of the mean flow, (B.13)–(B.17) become:

$$\mathcal{W}_{1s}' = \frac{1}{S_c}\int m_s' dq \tag{B.19}$$

$$\begin{aligned}\mathcal{F}_{1s}' = {} & \frac{1}{S_c}\left\{(\bar{u}_s-\bar{u})\int m_{sg}'dq + (\bar{u}_{ls}-\bar{u})\int m_{sl}'dq\right\} \\ & + \frac{1}{S_c}\left\{(u_s'-u')\int \bar{m}_{sg}dq + (u_{ls}'-u_l')\int \bar{m}_{sl}dq\right\}\end{aligned} \tag{B.20}$$

$$\begin{aligned}\mathcal{Q}_{1s}' = {} & \frac{1}{S_c}\left\{(\bar{h}_{0s}-\bar{e}_0)\int m_{sg}'dq + (\bar{e}_{l0s}-\bar{e}_{l0})\int m_{sl}'dq + C_v\bar{T}\int m_{sg}'dq\right\} \\ & + \frac{1}{S_c}\left\{(h_{0s}'-e_0')\int \bar{m}_{sg}dq + (e_{l0s}'-e_{l0}')\int \bar{m}_{sl} + C_vT'\int \bar{m}_{sg}dq\right\}\end{aligned} \tag{B.21}$$

$$\begin{aligned}\mathcal{P}_{1s}' = {} & \frac{R}{C_v}\frac{1}{S_c}\left\{(\bar{h}_{0s}-\bar{e}_0)\int m_{sg}'dq + (\bar{e}_{l0s}-\bar{e}_{l0})\int m_{sl}'dq + C_v\bar{T}\int m_{sg}'dq\right\} \\ & + \frac{R}{C_v}\frac{1}{S_c}\left\{(h_{0s}'-e_0')\int \bar{m}_{sg}dq + (e_{l0s}'-e_{l0}')\int \bar{m}_{sl} + C_vT'\int \bar{m}_{sg}dq\right\}\end{aligned} \tag{B.22}$$

$$\mathcal{S}'_{1s} = \frac{1}{p}Q'_{1s} - \frac{p}{\rho}\mathcal{W}'_{1s} \tag{B.23}$$

The procedure for forming the wave equation in the one-dimensional approximation is the same as that week in Chapter 4, with the results valid to second order in small quantities

$$\frac{1}{S_c}\frac{\partial}{\partial x}\left(S_c\frac{\partial p'}{\partial x}\right) - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h_1 \tag{B.24}$$

$$\frac{\partial p'}{\partial x} = -f_1 \quad (x = 0, L) \tag{B.25}$$

and

$$\begin{aligned} h_1 = &-\bar{\rho}\frac{1}{S_c}\frac{\partial}{\partial x}\left(S_c\frac{\partial \bar{u}u'}{\partial x}\right) + \frac{\bar{u}}{\bar{a}^2}\frac{\partial^2 p'}{\partial t\partial x} + \frac{\bar{\gamma}}{\bar{a}^2}\frac{\partial p'}{\partial t}\frac{1}{S_c}\frac{\partial}{\partial x}(S_c\bar{u}) \\ &- \frac{1}{S_c}\frac{\partial}{\partial x}S_c\left(\bar{\rho}u'\frac{\partial u'}{\partial x} + \rho'\frac{\partial u'}{\partial t}\right) \\ &+ \frac{1}{\bar{a}^2}\frac{\partial}{\partial x}\left(u'\frac{\partial p'}{\partial x}\right) + \frac{\bar{\gamma}}{\bar{a}^2}\frac{\partial p'}{\partial t}\frac{1}{S_c}\frac{\partial}{\partial x}(S_c u') \\ &+ \frac{1}{S_c}\frac{\partial}{\partial x}S_c(\mathcal{F}'_1 + \mathcal{F}'_{1s}) - \frac{1}{\bar{a}^2}\frac{\partial(\mathcal{P}'_1 + \mathcal{P}'_{1s})}{\partial t} \end{aligned} \tag{B.26}$$

$$f_1 = \bar{\rho}\frac{\partial u'}{\partial t} + \bar{\rho}\frac{\partial}{\partial x}(\bar{u}u') + \bar{\rho}u'\frac{\partial u'}{\partial x} + \rho'\frac{\partial u'}{\partial t} - (\mathcal{F}'_1 + \mathcal{F}'_{1s}) \tag{B.27}$$

The normal modes for one-dimensional motions are found as the solutions to

$$\frac{1}{S_c}\frac{d}{dx}\left(S_c\frac{d\psi_l}{dx}\right) + k_l^2\psi_l = 0 \tag{B.28}$$

$$\frac{d\psi_l}{dx} = 0 \qquad (z = 0,\ L) \tag{B.29}$$

and the basic expansions for the acoustic field are

$$p'(x,t) = \bar{p}\sum_{j=1}^{M}\eta_j(t)\psi_j(x) \tag{B.30}$$

$$u'(x,t) = \sum_{j=1}^{M}\frac{\dot{\eta}_j(t)}{\bar{\gamma}k_j^2}\frac{d\psi_j(x)}{dx} \tag{B.31}$$

Orthogonality of the normal modes is expressed as

$$\int_0^L \psi_j \psi_l S_c dx = E_l^2 \delta_{jl} \tag{B.32}$$

$$E_l^2 = \int_0^L \psi_l^2 S_c dx \tag{B.33}$$

With a procedure similar to that leading to equation (4.36), we find the equations for the amplitudes:

$$\frac{d^2\eta_l}{dt^2} + \omega_l^2 \eta_l = F_l \tag{B.34}$$

$$F_l = -\frac{\bar{a}^2}{\bar{p}E_l^2}\left\{\int_0^L h_1 \psi_l S_c dx + [f_1 \psi_l S_c]_0^L\right\} \tag{B.35}$$

Note that F_l here is not the one-dimensional form of the force found from (A.32). The current context will distinguish the two symbols.

Development of the principal basic results for stability treated with the approximation of one-dimensional motions is covered in Section 6.9.