ANNEX C

The Acoustic Boundary Layer

Consider an impermeable rigid wall exposed to a flow of gas oscillating parallel to the surface but having no average motion. The gas is viscous and heat conducting. At the surface, the velocity is zero and the temperature is maintained constant. Far from the surface all flow properties oscillate sinusoidally and the motion is isentropic. The values of the velocity and temperature vary from their fixed values at the surface to their sinusoidal values far from the surface through a two-dimensional boundary layer. For the following analysis we assume that there is no communication parallel to the surface. That is, we assume that flow within the boundary layer at any position x depends only on the conditions imposed locally by the external flow. This assumption is valid because the flow is oscillating with no mean velocity. Figure C.1 is a sketch of the flow.

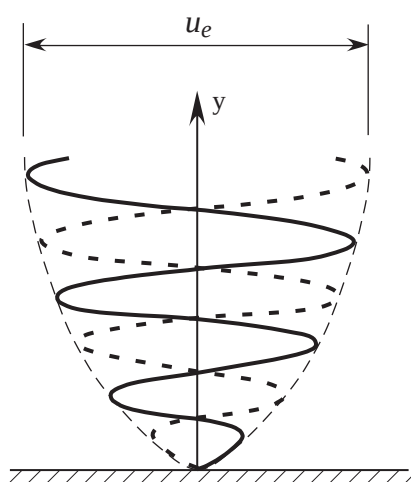

FIGURE C.1. Sketch of the flow in an acoustic boundary layer showing the velocity field at two instants separated by one-half period.

We assume that the oscillations are sustained by some external actions far from the region under consideration. Moreover, for small viscosity—high Reynolds number based on the amplitude of the velocity outside the boundary layer, and the 'thickness' of the layer, found as part of the solution—the boundary layer is thin. We may assume this to be the case, and check the assumption later. Then according to the principles of boundary layer theory, we take the pressure constant in the direction normal to the surface.

The linearized mass, momentum and energy equations, (5.7)–(5.9) written for incompressible flow with viscous stresses and heat combustion included, are

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla\cdot\mathbf{u}' &= 0 \\ \bar{\rho}\frac{\partial \mathbf{u}'}{\partial t} + \nabla p' &= \nabla\cdot\boldsymbol{\tau}'_v \\ \bar{\rho}\frac{\partial e'}{\partial t} + \bar{\rho}\nabla\cdot\mathbf{u}' &= -\nabla\cdot\mathbf{q}' \end{aligned} \tag{C.1)a,b,c}$$

It is convenient for this problem to replace the energy by the enthalpy, $h = e + p/\rho$, so

$$\frac{\partial e}{\partial t} = \frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{p}{\rho^2}\frac{\partial \rho}{\partial t}$$

For incompressible flow, ρ' is zero and (C.1)a gives $\nabla\cdot\mathbf{u}' = 0$. By definition, $dh = C_p dT$, and the energy equation (C.1)c becomes

$$\bar{\rho}C_p\frac{\partial T'}{\partial t} = \frac{\partial p'}{\partial t} - \nabla\cdot\mathbf{q}' \tag{C.2}$$

Under the conditions supposed here, the contributions from viscous and heat conduction are (see, e.g., Schlicting 1960):

$$\nabla\cdot\boldsymbol{\tau}'_v = \mu\frac{\partial^2 u'}{\partial y^2} \quad ; \qquad \nabla\cdot\mathbf{q}' = -\lambda_c\frac{\partial^2 T'}{\partial y^2} \tag{C.3}$$

Hence the momentum and energy equations for this problem are[1]

$$\begin{aligned} \bar{\rho}\frac{\partial u'}{\partial t} &= \mu\frac{\partial^2 u'}{\partial y^2} + \bar{\rho}\frac{\partial u'_e}{\partial t} \\ \bar{\rho}C_p\frac{\partial T'}{\partial t} &= \lambda_c\frac{\partial^2 T'}{\partial y^2} + \frac{\partial p'_e}{\partial t} \end{aligned} \tag{C.4)a,b}$$

It is an important feature of this formulation that the coordinate x along the surface appears only parametrically in $u'_e(x;t)$ and $p'_e(x;t)$. Later we discuss application and interpretation of this feature.

We assume sinusoidal time dependence and write

$$u' = \hat{u}e^{-i\omega t} \quad ; \qquad u'_e = \hat{u}_e e^{-i\omega t} \quad ; \qquad p'_e = \hat{p}_e e^{-i\omega t}$$

Equations (C.4)a,b become

$$\begin{aligned} \frac{d^2\hat{u}}{dy^2} + i\frac{2}{\delta^2}\hat{u} &= +i\frac{2}{\delta^2}\hat{u}_e \\ \frac{d^2}{dy^2}\left(\frac{\hat{T}}{\bar{T}_e}\right) + i\frac{2Pr}{\delta^2}\left(\frac{\hat{T}}{\bar{T}_e}\right)\hat{u} &= +i\frac{2Pr}{\delta^2}\frac{\gamma-1}{\gamma}\left(\frac{\hat{p}}{\bar{p}_e}\right) \end{aligned} \tag{C.5)a,b}$$

where the characteristic thickness of the boundary layer is

$$\delta = \sqrt{\frac{2\nu}{\omega}} \tag{C.6}$$

[1]We have replaced the pressure gradient in the momentum equation by the rate of change of acoustic momentum. That follows because the pressure doesn't change through this layer and therefore obeys the acoustic momentum equation for the external flow.

and $\nu = \mu/\bar{\rho}$ is the kinematic viscosity and $Pr = C_p\mu/\lambda_c$ is the Prandtl number. The solutions to the boundary layer equations can be written as sums of homogeneous and particular solutions

$$\begin{aligned} \hat{u} &= \hat{u}_h + \hat{u}_e \\ \hat{T} &= \hat{T}_h + \frac{\gamma-1}{\gamma}\left(\frac{\bar{T}_e}{\bar{p}_e}\right)\hat{p}_e \end{aligned} \qquad \text{(C.7)a,b}$$

and $\hat{u}_h$, $\hat{T}_h$ satisfy

$$\begin{aligned} \frac{d^2\hat{u}_h}{dy^2} + i\frac{2}{\delta^2}\hat{u} &= 0 \\ \frac{d^2\hat{T}_h}{dy^2} + i\frac{2Pr}{\delta^2}\hat{T}_h &= 0 \end{aligned} \qquad \text{(C.8)a,b}$$

The solutions (C.7)a,b must satisfy the boundary conditions

$$\begin{aligned} y \to \infty : &\qquad \hat{u} = \hat{u}_e \quad , \qquad \hat{T} = \hat{T}_e \\ y = 0 : &\qquad \hat{u} = 0 \quad ; \qquad \hat{T} = 0 \end{aligned} \qquad \text{(C.9)}$$

Homogeneous solutions to (C.8)a,b are

$$\hat{u}_h = c_1 e^{-\lambda y} \quad ; \qquad \hat{T}_h = c_2 e^{-\sqrt{Pr}y} \qquad \text{(C.10)a,b}$$

with λ satisfying the equation

$$\lambda^2 - i\frac{2}{\delta^2} = 0 \qquad \text{(C.11)}$$

Hence $\lambda = \sqrt{2i}\frac{1}{\delta}$,

$$\lambda = \frac{1}{\delta}(1-i) \qquad \text{(C.12)}$$

The value of λ having positive real part is chosen so $\hat{u}_h \to 0$ as $y \to \infty$; replacing (C.10)a,b in (C.7)a,b, we have

$$\begin{aligned} \hat{u} &= c_1 e^{-\lambda y} + \hat{u}_e \\ \hat{T} &= c_2 e^{-\sqrt{Pr}y} + \frac{\gamma-1}{\gamma}\left(\frac{\bar{T}_e}{\bar{p}_e}\right)\hat{p} \end{aligned} \qquad \text{(C.13)a,b}$$

To satisfy the boundary conditions at the surface, c_1 and c_2 are assigned the values

$$c_1 = -\hat{u}_e \quad ; \qquad c_2 = -\frac{\gamma-1}{\gamma}\left(\frac{\bar{T}_e}{\bar{p}_e}\right)$$

Hence the solutions for the distributions of velocity and temperature within the boundary layer are

$$\begin{aligned} \hat{u} &= \hat{u}_e\left[1 - e^{-\lambda y}\right] \\ \hat{T} &= \hat{T}_e\left[1 - e^{-\lambda\sqrt{Pr}y}\right] \end{aligned} \qquad \text{(C.14)a,b}$$

where we have assumed isentropic motion in the external flow:

$$\frac{\hat{T}_e}{\bar{T}_e} = \frac{\gamma-1}{\gamma}\frac{\hat{p}_e}{\bar{p}_e} \quad \text{and} \quad \hat{u}_e = \frac{-i}{\bar{\rho}\omega}\frac{d\hat{p}_e}{dx}$$

Although we have assumed the flow to be incompressible in the above analysis, the boundary layer is usually referred to as the 'acoustic boundary layer'. The flow near the surface is dominated by inertial and viscous forces so compressible effects are ignored. However, the result can (and usually is) used with a compressible, i.e. acoustic, field external to the boundary layer.

Mechanical energy of motion is dissipated within the acoustic boundary layer, causing acoustic waves incident on a wall to be attenuated. That process is analyzed in Section 5.9. For the calculations we need a formula for the rate of dissipation of energy within the acoustic boundary layer. The general form is (A.63) with the sources given by (A.66) and the definitions (A.67) and (A.69) for $\mathcal{W}$ and $\mathcal{Q}$:

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{T}\left[\mathcal{Q} - \frac{p}{\rho^2}\mathcal{W}\right] \tag{C.15}$$

For the conditions here, the only terms surviving in $\mathcal{Q}$ and $\mathcal{W}$ are those due to viscous stresses and heat conduction; then $\mathcal{W} = 0$ and

$$\mathcal{Q} = \boldsymbol{\tau}_v \cdot \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q}$$

For the one-dimensional flow in the acoustic layer,

$$\begin{aligned} \boldsymbol{\tau}_v \cdot \nabla \cdot \mathbf{u} &= \mu \left(\frac{\partial u}{\partial y}\right)^2 \\ -\nabla \cdot \mathbf{q} &= \frac{\partial}{\partial y}\left(\lambda_c \frac{\partial T}{\partial y}\right) \end{aligned} \tag{C.16)a,b}$$

and

$$\frac{1}{T}\mathcal{Q} = \frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{1}{T}\frac{\partial}{\partial y}\left(\lambda_c \frac{\partial T}{\partial y}\right)$$

The last term can be rewritten and for this flow (C.15) becomes

$$\frac{\partial s}{\partial t} = \left[\frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\lambda_c}{T^2}\left(\frac{\partial}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left(\frac{\lambda_c}{T}\frac{\partial T}{\partial y}\right) \tag{C.17}$$

To understand this result, consider an element of the flow extending between the planes y_1 and y_2 and having unit area. The right hand side of (C.17) represents the rate of entropy production per unit volume associated with this element. There are two contributions which can be interpreted with the following calculation. First integrate (C.17) over the volume of the element; for one dimensional flow, only the integral over y remains:

$$\int_{y_1}^{y_2} \frac{\partial s}{\partial t} dy = \int_{y_1}^{y_2} \left[\frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\lambda_c}{T^2}\left(\frac{\partial T}{\partial y}\right)^2\right] dy + \left[\frac{\lambda_c}{T^2}\frac{\partial T}{\partial y}\right]_{y_1}^{y_2} \tag{C.18}$$

This result represents the total entropy production associated with the volume element. The bracketted terms under the integral must be interpreted as the entropy production per unit volume arising from dissipation of energy. The first term is due to action of viscous stresses and the second is due to conduction of heat.

The last term in (C.18) is the net entropy production due to conduction of heat through the two faces of the element at $y = y_1, y_2$. This part of the entropy production must be assigned to the environment of the element, for the following reason. Now apply (C.18) to the acoustic boundary layer with $y_1 = 0$ at the wall, and $y_2 \to \infty$ so the entire layer is included in the integration. Then $\partial T/\partial y = 0$ in y_2 and (C.18) is

$$\int_0^\infty \frac{\partial s}{\partial t} dy = \int_0^\infty \left[\frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\lambda_c}{T^2}\left(\frac{\partial T}{\partial y}\right)^2\right] dy - \left(\frac{\lambda_c}{T^2}\frac{\partial T}{\partial y}\right)_{y=0}$$

Suppose that the temperature increases with y. Then heat is transferred from the element to the wall, causing a loss of entropy of the element and an increase of entropy of the wall. That heat flow is non-zero because of the temperature rise due to energy dissipation within the element. Therefore, if we are concerned

only with the consequences of processes within the element, we may drop this term. Hence the formula for the rate of entropy production per unit volume within the element is

$$\frac{\partial s}{\partial t} = \frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\lambda_c}{T^2}\left(\frac{\partial T}{\partial y}\right)^2 \tag{C.19}$$