ANNEX D

Accounting for Waves of Vorticity and Entropy

In Chapters 3 and 4, the equations of motion have been written in dimensionless variables, mainly as the basis for the two-parameter expansion. It is often preferable to use dimensionless variable, for example, in modeling some of the physical processes treated in special problems. In this annex, the main results of Chapter 3 are reproduced, and extended to show explicitly the formal consequences of including unsteady vorticity and entropy in the method of spatial averaging developed in Chapter 4.

The basis for the calculations in the set of equations[1] (A.59)–(A.64)

$$
\begin{aligned}
\frac{D\rho}{Dt} &= -\rho\nabla\cdot\mathbf{u} + \mathcal{W} \\
\rho\frac{D\mathbf{u}}{Dt} &= -\nabla p + \boldsymbol{\mathcal{F}} \\
\rho C_v\frac{DT}{Dt} &= -p\nabla\cdot\mathbf{u} + \mathcal{Q} \\
\frac{Dp}{Dt} &= -\gamma p\nabla\cdot\mathbf{u} + \mathcal{P} \\
\frac{Ds}{Dt} &= \frac{1}{T}\mathcal{S} \\
p &= \rho RT
\end{aligned}
\qquad \text{(D.1)a-f}
$$

Following the procedure explained in Chapter 3, expansion of (D.1)a-f to third order in the fluctuations. Problem III defined in Section 3.3.3 leads to the equations governing the unsteady field. To simplify the

[1] Here bold symbols for the mass-averaged fluid properties $\mathbf{R}$, $\mathbf{C}_p$, $\mathbf{C}_v$ and $\boldsymbol{\gamma}$ are replaced by R, C_p, C_v and γ

results we assume that the average values $\bar{\rho}$, $\bar{p}$, $\bar{T}$ are constant and uniform to give:

$$\frac{\partial \rho'}{\partial t} + \bar{\rho}\nabla\cdot\mathbf{u}' = -[\bar{\mathbf{u}}\cdot\nabla\rho' + \mathbf{u}'\cdot\nabla\bar{\rho} + \rho'\nabla\cdot\bar{\mathbf{u}}] - [\mathbf{u}'\cdot\nabla\rho' + \rho'\nabla\cdot\mathbf{u}'] + \mathcal{W}'$$

$$\bar{\rho}\frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = -[\bar{\rho}(\bar{\mathbf{u}}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\bar{\mathbf{u}})] - \left[\bar{\rho}\mathbf{u}'\cdot\nabla\mathbf{u}' + \rho'\frac{\partial \mathbf{u}'}{\partial t}\right] - [\rho'\mathbf{u}'\cdot\nabla\mathbf{u}'] + \boldsymbol{\mathcal{F}}'$$

$$\begin{aligned}\bar{\rho}C_v\frac{\partial T'}{\partial t} + \bar{p}\nabla\cdot\mathbf{u}' = &-[\bar{\rho}C_v(\bar{\mathbf{u}}\cdot\nabla T' + \mathbf{u}'\cdot\nabla\bar{T}) + p'\nabla\cdot\bar{\mathbf{u}}] - \left[\rho'C_v\frac{\partial T'}{\partial t} + \bar{\rho}C_v\mathbf{u}'\cdot\nabla T' + p'\nabla\cdot\mathbf{u}'\right]\\ &- [\rho'C_v\mathbf{u}'\cdot\nabla T'] + \mathcal{Q}'\end{aligned}$$

(D.2)a-f

$$\frac{\partial p'}{\partial t} + \gamma\bar{p}\nabla\cdot\mathbf{u}' = -[\bar{\mathbf{u}}\cdot\nabla p' + \gamma p'\nabla\cdot\bar{\mathbf{u}}'] - [\mathbf{u}'\cdot\nabla p' + \gamma p'\nabla\cdot\mathbf{u}'] + \mathcal{P}'$$

$$\begin{aligned}\bar{\rho}\bar{T}\frac{\partial s'}{\partial t} = &-[\bar{\rho}T'(\bar{\mathbf{u}}\cdot\nabla s' + \mathbf{u}'\cdot\nabla\bar{s})] - \left[\left(\bar{\rho}T' + \rho'\bar{T}\right)\frac{\partial s'}{\partial t} + \bar{\rho}\bar{T}\mathbf{u}'\cdot\nabla s'\right]\\ &- [\rho'T'\frac{\partial s'}{\partial t} + \left(\bar{\rho}\bar{T} + \rho'\bar{T}\right)\mathbf{M}'\cdot\nabla s'] + \mathcal{S}'\end{aligned}$$

$$p' = [R(\rho'\bar{T} + \bar{\rho}T')] + [R\rho'T']$$

Combination of (D.2)b and (D.2)d gives the nonlinear wave equation for the pressure fluctuations and its boundary condition

$$\begin{aligned}\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} &= h\\ \hat{n}\cdot\nabla p' &= -f\end{aligned}$$

(D.3)a,b

with

$$h = -\bar{\rho}\nabla\cdot[\{[\mathbf{u}]\}_1 + \{\mathbf{u}\}_2 + \{\mathbf{u}\}_3] + \frac{1}{\bar{a}^2}\frac{\partial}{\partial t}[\{[p]\}_1 + \{p\}_2 + \{p\}_3] + \nabla\cdot\boldsymbol{\mathcal{F}}' - \frac{1}{\bar{a}^2}\frac{\partial p'}{\partial t}$$

(D.4)a,b

$$f = -\bar{\rho}\frac{\partial \mathbf{u}'}{\partial t}\cdot\hat{\mathbf{n}} + \bar{\rho}\hat{\mathbf{n}}\cdot[\{[\mathbf{u}]\}_1 + \{\mathbf{u}\}_2 + \{\mathbf{u}\}_3] - \boldsymbol{\mathcal{F}}'\cdot\hat{\mathbf{n}}$$

The brackets are defined similarly to those in Annex A.6:

$$\{[\mathbf{u}]\}_1 = \bar{\rho}(\bar{\mathbf{u}}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\bar{\mathbf{u}}) \qquad \{[p]\}_1 = \bar{\mathbf{u}}\cdot\nabla p' + \gamma p'\nabla\cdot\bar{\mathbf{u}}$$

$$\begin{aligned}\{\mathbf{u}\}_2 = &\rho'\frac{\partial \mathbf{u}'}{\partial t} + \bar{\rho}\mathbf{u}'\cdot\nabla\mathbf{u}'\\ &+ \rho'(\bar{\mathbf{u}}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\bar{\mathbf{u}}) \qquad \{p\}_2 = \mathbf{u}'\cdot\nabla p' + \gamma p'\nabla\cdot\mathbf{u}'\end{aligned}$$

(D.5)a–e

$$\{\mathbf{u}\}_3 = \rho'\mathbf{u}'\cdot\nabla\mathbf{u}'$$

A system of oscillator equations is formed as in Chapter 4. Expand the pressure fluctuation in normal modes:

$$p'(\mathbf{r},t) = \bar{p}\sum_{m=0}^{M}\eta_m(t)\psi_m(\mathbf{r}) \tag{D.6}$$

Spatial averaging leads to the result (4.36) with n replacing N,

$$\ddot{\eta}_n + \omega_n^2\eta_n = F_n \tag{D.7}$$

and

$$F_n = -\frac{\bar{a}^2}{\bar{p}E_n^2}\left\{\int h\psi_n dV + \oiint f\psi_n dS\right\} \tag{D.8}$$

With some use of familiar vector identities, for h and f given by (D.6)a,b, F_n can eventually be written

$$\begin{aligned}-\frac{\bar{p}E_n^2}{\bar{a}^2}F_n = \bar{\rho}I_1 + \frac{1}{\bar{a}^2}I_2 + \bar{\rho}I_3 + \frac{1}{\bar{a}^2}I_4 + \bar{\rho}I_5 + \oiint\bar{\rho}\frac{\partial\mathbf{u}}{\partial t}\cdot\hat{\mathbf{n}}dS \\ - \int \boldsymbol{\mathcal{F}}'\cdot\nabla\psi_n dV + \frac{1}{\bar{a}^2}\int\frac{\partial\mathcal{P}'}{\partial t}\psi_n dV\end{aligned} \tag{D.9}$$

where

$$\begin{aligned}
I_1 &= \int(\bar{\mathbf{u}}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\bar{\mathbf{u}})\cdot\nabla\psi_n dV \\
I_2 &= \frac{\partial}{\partial t}\int(\gamma p'\nabla\cdot\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\nabla p')\psi_n dV \\
I_3 &= \int(\mathbf{u}'\cdot\nabla\mathbf{u}' + \frac{\rho'}{\bar{\rho}}\frac{\partial\mathbf{u}'}{\partial t})\cdot\nabla\psi_n dV \\
I_4 &= \frac{\partial}{\partial t}\int(\gamma p'\nabla\cdot\mathbf{u}' + \mathbf{u}'\cdot\nabla p')\psi_n dV \\
I_5 &= \int\frac{\rho'}{\bar{\rho}}(\mathbf{u}'\cdot\nabla\mathbf{u}')\cdot\nabla\psi_n dV
\end{aligned} \tag{D.10)a-e}$$

It is important that both the steady and unsteady velocity fields have *not* been assumed to be irrotational. Note also that in (D.9) there is only one surface term, involving the acceleration $\frac{\partial\mathbf{u}'}{\partial t}$. However, for irrotational acoustic motions some of the integrands combine to form a divergence which by Gauss' theorem leads to a surface integral representing convection of mechanical energy throughout the surface.

These results establish the formal basis for investigating influences of vorticity and entropy waves on an acoustic field in a chamber. We emphasize again that the expansion (D.6) of the pressure field is only the zeroth order approximation. The oscillator equations govern the time-evolution of the amplitudes of the chosen basis functions, the mode shapes, but the spatial distributions to higher order require further computations, as explained in Section 4.6. The dependent variables in h and f are not to this point restricted. Hence in particular, following the idea discussed in Section 4.1 that generally a disturbance in a compressible flowing medium may be synthesized of three modes of propagation, we write the variables as sums of three

parts labeled $(\quad)_a$ for acoustic, $(\quad)_\Omega$ for vorticity, $(\quad)_s$ for entropy; the second column of Table D.1 shows the defining properties of the contributions from the three modes in their zeroth approximation.

TABLE D.1. Definition of Splitting a Disturbance in the Three Modes of Propagation (Chu and Kovasznay, 1958).

	Splitting Variables	Basic Properties
Pressure	$p' = p'_a + p'_\Omega + p'_s$	$p'_a \neq 0 \quad ; \quad \Omega'_\Omega = \Omega'_s = 0$
Vorticity	$\Omega' = \Omega'_\Omega + \Omega'_s + \Omega'_a$	$\Omega'_\Omega \neq 0 \quad ; \quad \Omega'_s = \Omega'_a = 0$
Entropy	$s' = s'_s + s'_a + s'_\Omega$	$s'_s \neq 0 \quad ; \quad s'_a = s'_\Omega = 0$

The zeroth approximations (small amplitude; no mean flow) to the flow variables in the three modes are given in Table D.2.

TABLE D.2. Zeroth Approximation for the Flow Variables in the Three Modes.

	Pressure	Velocity
Acoustic Mode	$p'_a = \bar{p} \sum_{m=1}^{M} \eta_m(t)\psi_m(\mathbf{r})$	$\mathbf{u}'_a = \sum_{m=1}^{M} \frac{\dot{\eta}_m}{\gamma k_m^2} \nabla\psi_m$
Vorticity Mode	$p'_\Omega = 0$	$\nabla \times \mathbf{u}'_\Omega = 0 \; ; \quad \nabla \cdot \mathbf{u}'_\Omega = 0$
Entropy Mode	$p'_s = 0$	$\nabla \times \mathbf{u}'_s = 0 \; ; \quad \nabla \cdot \mathbf{u}'_s = \frac{1}{c_p}\frac{\partial s'}{\partial t}$

Thus the total velocity fluctuation **to zeroth order** is

$$\mathbf{u}'(\mathbf{r};t) = \mathbf{u}'_a + \mathbf{u}'_\Omega + \mathbf{u}'_s \tag{D.11}$$

The density fluctuation contains a contribution from the entropy fluctuation (related to the temperature); from the formula for the entropy of a perfect gas,

$$s - s_0 = c_p \log \frac{(p/p_0)^{1/\gamma}}{(\rho/\rho_0)} \tag{D.12}$$

We find, to second order in the pressure fluctuations,

$$\frac{\rho'}{\bar{\rho}} = \left[\frac{1}{\gamma}\frac{p'}{\bar{p}} - \frac{s'}{c_p}\right] - \frac{\gamma - 1}{2\gamma^2}\left(\frac{p'}{\bar{p}}\right)^2 \tag{D.13}$$

The next step is substitution of (D.11) in the integrals (D.10)a-e and rearrangement to give the formulas for the first and second order contributions:

$$\begin{aligned} I_1 &= I_1^a + I_1^\Omega + I_1^s \\ I_2 &= I_2^a \\ I_3 &\rightarrow I_3^{ij} \\ I_4 &= I_4^{aa} \end{aligned} \tag{D.14)a-d}$$

The elements I_3^{ij} forming $I_3 = \sum_{j=1}^{3}\sum_{i=1}^{3} I_3^{ij}$ are

$$I_3^{aa} = \int \left(\mathbf{u}'_a \cdot \nabla \mathbf{u}'_a + \frac{\rho'_a}{\bar{\rho}} \frac{\partial \mathbf{u}'_a}{\partial t} \right) \cdot \nabla \psi_n dV \ ; \quad I_3^{a\Omega} = \int (\mathbf{u}'_a \cdot \nabla \mathbf{u}'_\Omega) \cdot \nabla \psi_n dV \ ;$$
$$I_3^{as} = \int \left(\mathbf{u}'_a \cdot \nabla \mathbf{u}'_s + \frac{\rho'_a}{\bar{\rho}} \frac{\partial \mathbf{u}'_s}{\partial t} \right) \cdot \nabla \psi_n dV$$

$$I_3^{\Omega a} = \int (\mathbf{u}'_\Omega \cdot \nabla \mathbf{u}'_a) \cdot \nabla \psi_n dV \ ; \quad I_3^{\Omega\Omega} = \int (\mathbf{u}'_\Omega \cdot \nabla \mathbf{u}'_\Omega) \cdot \nabla \psi_n dV \ ;$$
$$I_3^{\Omega s} = \int (\mathbf{u}'_\Omega \cdot \nabla \mathbf{u}'_s) \cdot \nabla \psi_n dV \tag{D.15}$$

$$I_3^{sa} = \int \left(\mathbf{u}'_s \cdot \nabla \mathbf{u}'_a + \frac{\rho'_s}{\bar{\rho}} \frac{\partial \mathbf{u}'_a}{\partial t} \right) \cdot \nabla \psi_n dV \ ; \quad I_3^{s\Omega} = \int (\mathbf{u}'_s \cdot \nabla \mathbf{u}'_\Omega) \cdot \nabla \psi_n dV \ ;$$
$$I_3^{ss} = \int \left(\mathbf{u}'_s \cdot \nabla \mathbf{u}'_s + \frac{\rho'_s}{\bar{\rho}} \frac{\partial \mathbf{u}'_s}{\partial t} \right) \cdot \nabla \psi_n dV$$

Equation (D.13) gives the two parts of the density fluctuation:

$$\frac{\rho'_a}{\bar{\rho}} = \frac{1}{\gamma} \frac{p'}{\bar{p}}$$
$$\frac{\rho'_s}{\bar{\rho}} = -\frac{s'}{c_p} \tag{D.16)a,b}$$

The remaining integrals in (D.14)a-d are defined as

$$I_1^a = \int [\nabla(\bar{\mathbf{u}} \cdot \mathbf{u}'_a) - \mathbf{u}'_a \times (\nabla \times \bar{\mathbf{u}})] \cdot \nabla \psi_n dV$$
$$I_1^\Omega = \int [\nabla(\bar{\mathbf{u}} \cdot \mathbf{u}'_\Omega) - \mathbf{u}'_\Omega \times (\nabla \times \bar{\mathbf{u}}) - \bar{\mathbf{u}} \times \Omega'] \cdot \nabla \psi_n dV \tag{D.17)a,b}$$
$$I_1^s = \int [\nabla(\bar{\mathbf{u}} \cdot \mathbf{u}'_s) - \mathbf{u}'_a \times (\nabla \times \bar{\mathbf{u}}) - \bar{\mathbf{u}} \times (\nabla \times \mathbf{u}'_s)] \cdot \nabla \psi_n dV$$

and

$$I_2 = I_2^{aa} = \frac{\partial}{\partial t} \int (\gamma p'_a \nabla \cdot \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla p'_a) \psi_n dV$$
$$I_4 = I_4^{aa} = \frac{\partial}{\partial t} \int (\gamma p'_a \nabla \cdot \mathbf{u}'_a + \mathbf{u}'_a \cdot \nabla p'_a) \psi_n dV \tag{D.18)a,b}$$

Because the integrals I_i are additive in (D.9), we can set down the parts which involve only acoustic, vorticity or entropy fluctuations, and the interactions; it is more physically appealing to write F_n, equation (D.9), as the sum:

$$-\frac{\bar{p} E_n^2}{\bar{a}^2} F_n = I_{aa} + I_{\Omega\Omega} + I_{ss} + I_{a\Omega} + I_{as} + I_{\Omega s}$$
$$- \int \boldsymbol{\mathfrak{F}}' \cdot \nabla \psi_n dV + \frac{1}{\bar{a}^2} \int \frac{\partial \mathcal{P}'}{\partial t} \psi_n dV \tag{D.19}$$

with the definition

$$I_{aa} = \bar{\rho} k_n^2 \int \bar{\mathbf{u}} \cdot \mathbf{u}'_a \psi_n dV - \bar{\rho} \int (\mathbf{u}'_a \times \nabla \times \bar{\mathbf{u}}) \cdot \nabla \psi_n dV + \frac{1}{\bar{a}^2} \frac{\partial}{\partial t} \int (\gamma p'_a \nabla \cdot \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \mathbf{u}'_a) \psi_n dV$$
$$+ \bar{\rho} I_3^{aa} + \frac{1}{\bar{a}^2} I_4^{aa} + \bar{\rho} \oiint \frac{\partial \mathbf{u}'}{\partial t} \cdot \hat{\mathbf{n}} \psi_n dV \tag{D.20}$$

Substitution of the acoustic approximations to the acoustic pressure and velocity (see Table D.2) leads eventually to the simpler formula for I_{aa}:

$$I_{aa} = \bar{\rho} \oiint \left[\frac{\partial \mathbf{u}'}{\partial t} \cdot \hat{\mathbf{n}} + \frac{1}{\gamma} (\bar{\mathbf{u}} \cdot \hat{\mathbf{n}}) \eta_n \psi_n \right] \psi_n dS + \bar{\rho} {\sum_{j=1}^{M}}' \frac{\dot{\eta}_j}{\gamma} \left(\frac{k_n^2}{k_j^2} + 1 \right) \int (\bar{\mathbf{u}} \cdot \nabla \psi_j) \psi_n dV$$
$$- \bar{\rho} \int (\mathbf{u}'_a \times \nabla \times \bar{\mathbf{u}}) \cdot \nabla \psi_n dV + \bar{\rho} I_3^{aa} + \frac{1}{\bar{a}^2} I_4^{aa} \tag{D.21}$$

Note that the surface integral contains the entire velocity fluctuation, $\mathbf{u}' = \mathbf{u}'_a + \mathbf{u}'_\Omega + \mathbf{u}'_s$.

Without modeling the relevant physical processes, the remaining integrals in (D.19) cannot be simplified further than their primitive definition written as the sums appearing in (D.9), the individual contributions being given in (D.15) and (D.17):

$$I_{\Omega\Omega} = \bar{\rho} \left\{ I_1^\Omega + I_3^{\Omega\Omega} \right\}$$
$$= \bar{\rho} \left\{ \int [\nabla(\bar{\mathbf{u}} \cdot \mathbf{u}'_\Omega) - \mathbf{u}'_\Omega \times (\nabla \times \bar{\mathbf{u}}) - \bar{\mathbf{u}} \times \Omega'] \cdot \nabla \psi_n dV + \int (\mathbf{u}'_\Omega \cdot \nabla \mathbf{u}'_\Omega) \cdot \nabla \psi_n dV \right\}$$

$$I_{ss} = \bar{\rho} \left\{ I_1^s + I_3^{ss} \right\}$$
$$= \bar{\rho} \left\{ \int [\nabla(\bar{\mathbf{u}} \cdot \mathbf{u}'_s) - \mathbf{u}'_s \times (\nabla \times \bar{\mathbf{u}}) - \bar{\mathbf{u}} \times (\nabla \times \mathbf{u}'_s)] \cdot \nabla \psi_n dV + \int \left[(\mathbf{u}'_s \cdot \nabla \mathbf{u}'_s) + \frac{\rho'_s}{\bar{\rho}} \frac{\partial \mathbf{u}'_s}{\partial t} \right] \cdot \nabla \psi_n dV \right\}$$

$$I_{a\Omega} = \bar{\rho} \left\{ I_3^{a\Omega} + I_3^{\Omega a} \right\}$$
$$= \bar{\rho} \int [\mathbf{u}'_a \cdot \nabla \mathbf{u}'_\Omega + \mathbf{u}'_\Omega \cdot \nabla \mathbf{u}'_a] \cdot \nabla \psi_n dV$$

$$I_{as} = \bar{\rho} \left\{ I_3^{as} + I_3^{sa} \right\}$$
$$= \bar{\rho} \int \left[\mathbf{u}'_a \cdot \nabla \mathbf{u}'_s + \mathbf{u}'_s \cdot \nabla \mathbf{u}'_a + \frac{\rho'_a}{\bar{\rho}} \frac{\partial \mathbf{u}'_s}{\partial t} + \frac{\rho'_s}{\bar{\rho}} \frac{\partial \mathbf{u}'_a}{\partial t} \right] \cdot \nabla \psi_n dV$$

$$I_{s\Omega} = \bar{\rho} \left\{ I_3^{s\Omega} + I_3^{\Omega s} \right\}$$
$$= \bar{\rho} \int [\mathbf{u}'_s \cdot \nabla \mathbf{u}'_\Omega + \mathbf{u}'_\Omega \cdot \nabla \mathbf{u}'_s] \cdot \nabla \psi_n dV$$

(D.22)a-e