ANNEX F

Basis Functions Satisfying Homogeneous Boundary Conditions

The purpose of this annex is to clarify an important point that has been misunderstood by several critics of the methods followed in this book. It's a difficulty whose source is readily identified but a bit of effort is required to resolve the matter. The issue is perhaps best captured by the general question:

> **How is a true solution satisfying inhomogeneous boundary conditions represented accurately by an expansion in basis functions satisfying homogeneous boundary conditions?**

In the problems treated in this book, we are concerned often with the unsteady pressure field governed by the inhomogeneous wave equation (3.53) with the inhomogeneous boundary condition (3.55),

$$\nabla^2 p' - \frac{1}{\bar{a}^2}\frac{\partial^2 p'}{\partial t^2} = h \tag{F.1}$$

$$\hat{n}\cdot\nabla p' = -f \tag{F.2}$$

The approximate method of solution has been based on expansion of the unsteady pressure and velocity fields expressed in the basis functions ψ_n taken as normal modes satisfying the problem

$$\nabla^2\psi_n + k_n^2\psi_n = 0 \tag{F.3}$$

$$\hat{n}\cdot\nabla\psi_n = 0 \tag{F.4}$$

Then the pressure and velocity fields are approximated by the classical forms with amplitudes $\eta_n(t)$ to be determined:

$$\tilde{p}'(\mathbf{r},t) = \bar{p}\sum_{n=1}^{N}\eta_n(t)\psi_n(\mathbf{r}) \tag{F.5}$$

$$\tilde{\mathbf{u}}'(\mathbf{r},t) = \sum_{n=1}^{N}\frac{\dot{\eta}_n(t)}{\gamma k_n^2}\nabla\psi_n(\mathbf{r}) \tag{F.6}$$

where N may become infinite; see Section 4.3.

With the representation (F.5), spatially averaging the difference between the actual problem. (F.1), (F.2), and the normal mode problem (F.3), (F.4), gives the equations (4.36) for the amplitudes $\eta_n(t)$:

$$\frac{d^2\eta_n}{dt} + \omega_n^2\eta_n = -\frac{\bar{a}^2}{\bar{p}}\frac{1}{E_n^2}\left\{\int h\psi_n dV + \oiint f\psi_n dS\right\} \tag{F.7}$$

For reasons explained in Section 4.6, we require in this book *only* the classical formulas for the pressure and velocity because we compute all quantities—notably the amplitudes η_n and, if necessary, the pressure—to first order in the expansion parameter $\bar{M}_r$, a measure of the mean flow Mach number. That is, with only a modest (but tedious) extension of the apparatus we have covered, we could calculate p' and $\mathbf{u}'$ to first

order,

$$p'(\mathbf{r},t) = \bar{p}\sum_{n=0}^{N}\eta_n(t)\psi_n(\mathbf{r})\left[1+\bar{M}_r\mathcal{P}(r,t)\right] \tag{F.8}$$

$$\mathbf{u}'(\mathbf{r},t) = \sum_{n=0}^{N}\frac{\dot{\eta}_n(t)}{\gamma k_n^2}\left[\nabla\psi_n(\mathbf{r})+\bar{M}_r\mathbf{U}(r,t)\right] \tag{F.9}$$

We will avoid the calculations necessary to prove that the approximate solution, the expansions (F.5) and (F.6) really do solve the problem (F.1), (F.2) to the order required. To carry through the 'proof' would require finding $\mathcal{P}$ and $\mathbf{U}$ which the developments in the main text have shown are not needed for useful results. Therefore we make the point with a much simpler model problem.

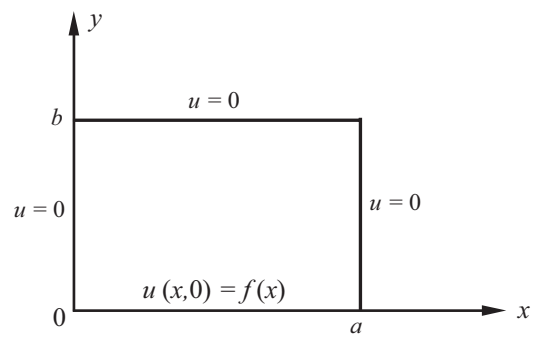

FIGURE F.1. The problem defined for determining $p(x,y)$.

The example was apparently first worked out by Friedman (1956, pp. 269–272). I learned the lesson, as well as useful explanation, from Professor D.S. Cohen, who clarified the problem in class notes. The following is an extended and paraphrased version of those notes as well as including what I learned from private conversations.

We consider the two-dimensional steady problem of finding $u(x,y)$ satisfying Laplace's equation with non-zero boundary condition only on the x-axis where $u(x,0) = f(x)$. Solution is found in the rectangle $0 < x < a$, $0 < y < b$, Figure F.1. Symbolically we have

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2} = 0 \qquad \begin{pmatrix}0<x<a\\0<y<b\end{pmatrix} \tag{F.10}$$

$$\left.\begin{aligned}p(x,0) &= f(x)\\ p(x,b) &= 0\end{aligned}\right\} \quad 0\le x\le a \tag{F.11}$$

$$\left.\begin{aligned}p(0,y) &= 0\\ p(a,y) &= 0\end{aligned}\right\} \quad 0\le y\le b \tag{F.12}$$

Corresponding to the zeroth order form (F.5) for the pressure field, we assume the solution to (F.10) which seemingly cannot satisfy the boundary condition (F.11),

$$p(x,y) = \sum_{n=1}^{\infty} c_n(x)\sin\frac{n\pi y}{b} \tag{F.13}$$

so

$$c_n(x) = \frac{2}{b}\int_0^b p(x,y)\sin\frac{n\pi y}{b}dy \tag{F.14}$$

The series (F.13) may be regarded as an expansion in the eigenfunctions $\sin\frac{n\pi y}{b}$ corresponding to ψ_n in (F.5). Spatial averaging of the governing equation (F.10) over y, weighted by the basic function, gives

$$\frac{2}{b}\int_0^b \frac{\partial^2 p}{\partial x^2}\sin\frac{n\pi y}{b}dy + \frac{2}{b}\int_0^b \frac{\partial^2 p}{\partial y^2}\sin\frac{n\pi y}{b}dy = 0$$

Then substitution of the assumed form (F.13) and integration of the second term by parts leads to

$$\frac{d^2c_n}{dx^2} + \frac{2}{b}\left\{\left[\frac{\partial p}{\partial y}(x,y)\sin\frac{n\pi y}{b}\right]_{y=0}^{y=b} - \frac{2n\pi}{b^2}\int_0^b \frac{\partial p}{\partial y}\cos\frac{n\pi y}{b}dy = 0\right.$$

The second term is zero and integration of the third term gives

$$\frac{d^2c_n}{dx^2} - \frac{2n\pi}{b^2}\left[p(x,y)\cos\frac{n\pi y}{b}\right]_{y=0}^{y=b} + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\int_0^b p(x,y)\sin\frac{n\pi y}{b}dy = 0$$

After substitution of the boundary condition on $y = 0$ in the first term, and the definition (F.14) in the last, the equation for $c_n(x)$ is

$$\frac{d^2c_n}{dx^2} - \left(\frac{n\pi}{b}\right)^2 c_n = -\frac{2n\pi}{b^2}f(x) \tag{F.15}$$

with the end conditions

$$c_n(0) = c_n(a) = 0 \tag{F.16}$$

Solution for $c_n(x)$ is conveniently constructed using a one-dimensional Green's function. Hildebrand (1952, pp. 388ff) has given a particularly clear discussion of the calculation which, with only minor changes, is directly applicable here. The Green's function $G(x|\xi)$ satisfies the same differential equation as $c_n(x)$ does except that the inhomogeneous right-hand side is non-zero at a single point ξ which lies in the range $a < \xi < b$,

$$\frac{d^2G(x|\xi)}{dx^2} - \kappa^2 G(x|\xi) = \delta(x-\xi) \tag{F.17}$$

where $\kappa = n\pi/b$ and $\delta(x-\xi)$ has the property,

$$\int_{x<\xi}^{x>\xi} \delta(x-\xi)d\xi = 1 \tag{F.18}$$

Without additional justification, we require G to have the four properties:

(i) G is composed of two parts, f and g, which because of the defining property (F.17) satisfy the same homogeneous equation:

$$\frac{d^2g}{dx^2} - \kappa^2 g = 0 \qquad (0 \le x < \xi)$$
$$\frac{d^2h}{dx^2} - \kappa^2 h = 0 \qquad (\xi < x \le a)$$

and

$$G = \begin{cases} g(x) & 0 \le x < \xi \\ h(x) & \xi < x \le a \end{cases} \tag{F.19}$$

(ii) G satisfies the prescribed homogeneous boundary conditions (F.16),

$$G(0|\xi) = G(a|\xi) = 0 \tag{F.20}$$

which imply

$$g(0) = 0\ , \quad h(a) = 0 \tag{F.21}$$

(iii) G is continuous at $x = \xi$,

$$g(\xi) = h(\xi) \tag{F.22}$$

(iv) the derivative of G has discontinuity equal to -1 at the point $x = \xi$:

$$\frac{dh}{dx} - \frac{dg}{dx} = -1 \quad (x = \xi)$$

Conditions (i) and (ii) are satisfied by the functions

$$\begin{aligned} g(x) &= A \sinh \kappa x \\ h(x) &= B \sinh \kappa (a - x) \end{aligned} \tag{F.23)a,b}$$

Conditions (iii) and (iv) then become

$$A \sinh \kappa\xi \quad - \quad B \sinh \kappa(a - \xi) \ = \ 0 \tag{F.24}$$

$$A \cosh \kappa\xi \quad + \quad B \cosh \kappa(a - \xi) \ = \ \frac{1}{\kappa} \tag{F.25}$$

The constants A and B are therefore

$$\begin{aligned} A &= \frac{1}{\Delta} \begin{vmatrix} 0 & -\sinh \kappa(a-\xi) \\ \frac{1}{\kappa} & +\cosh \kappa(a-\xi) \end{vmatrix} = \frac{1}{\kappa\Delta} \sinh \kappa(a - \xi) \\ B &= \frac{1}{\Delta} \begin{vmatrix} \sinh \kappa\xi & 0 \\ \cosh \kappa\xi & \frac{1}{\kappa} \end{vmatrix} = \frac{1}{\kappa\Delta} \sinh \kappa\xi \end{aligned} \tag{F.26)a,b}$$

where

$$\Delta = \sinh \kappa\xi \cosh \kappa(a - \xi) + \cosh \kappa\xi \sinh \kappa(a - \xi) = \sinh \kappa a \tag{F.27}$$

Thus $A = \sinh \kappa(a - \xi)/\kappa \sinh \kappa a$, $B = \sinh \kappa\xi/\kappa \sinh \kappa a$ and the Green's function for this problem follows from (F.19):

$$G(x|\xi) = \begin{cases} \frac{\sinh \kappa x \sinh \kappa(a-\xi)}{\kappa \sinh \kappa a} = \frac{\sinh \frac{n\pi x}{b} \sinh \frac{n\pi}{b}(a-\xi)}{\frac{n\pi}{b} \sinh n\pi \frac{a}{b}} & 0 \le x \le \xi \\ \frac{\sinh \kappa\xi \sinh \kappa(a-x)}{\kappa \sinh \kappa a} = \frac{\sinh \frac{n\pi\xi}{b} \sinh \frac{n\pi}{b}(a-x)}{\frac{n\pi}{b} \sinh n\pi \frac{a}{b}} & \xi \le x \le a \end{cases} \tag{F.28}$$

Solution for $c_n(x)$ is found in familiar fashion by combining (F.15) and (F.17). Multiply (F.17) by $c_n(x)$, (F.15) by $G(x|\xi)$, subtract the results and integrate over the range of the solution:

$$\begin{aligned} \int_0^a \left[c_n(x) \frac{d^2 G(x|\xi)}{dx^2} - G(x|\xi) \frac{d^2 c_n}{dx^2} \right] dx - \int_0^a \left[\kappa^2 c_n(x) G(x|\xi) - \left(\frac{n\pi}{b}\right)^2 G(x|\xi) c_n(x) \right] dx \\ = \int_0^a \left[c_n(x) \delta(x - \xi) + \frac{2n\pi}{b^2} f(x) G(x|\xi) \right] dx \end{aligned} \tag{F.29}$$

The first term is integrated by parts with the boundary conditions (F.16) and (F.20) applied:

$$\int_0^a \left[c_n(x)\frac{d^2G(x|\xi)}{dx^2} - G(x|\xi)\frac{d^2c_n(x)}{dx^2} \right] dx = \left[c_n\frac{dG}{dx} - G\frac{dc_n}{dx} \right]_0^a - \int_0^a \left[\frac{dc_n}{dx}\frac{dG}{dx} - \frac{dG}{dx}\frac{dc_n}{dx} \right] dx = 0$$

Because $\kappa = n\pi/b$, the integrand of the second integral in (F.29) is zero, giving

$$\int_0^a c_n(x)\delta(x-\xi)dx = -\frac{2n\pi}{b^2}\int_0^a f(x)G(x|\xi)dx$$

With the property (F.18),

$$c_n(\xi) = -\frac{2n\pi}{b^2}\int_0^a f(x)G(x|\xi)d\xi$$

Interchange x and ξ, and use the symmetry property which can be confirmed with (F.27)[1], $G(x|\xi) = G(\xi|x)$, the solution $c_n(x)$ is

$$c_n(x) = -\frac{2n\pi}{b^2}\int_0^a f(\xi)G(x|\xi)d\xi \tag{F.30}$$

Thus with (F.28) we have

$$\begin{aligned} c_n(x) &= -\frac{2n\pi}{b^2}\int_0^x f(\xi)\left[G(x|\xi)\right]_{0\le\xi\le x} d\xi - \frac{2n\pi}{b^2}\int_x^a f(\xi)\left[G(x|\xi)\right]_{x\le\xi\le a} d\xi \\ &= -\frac{2}{b}\frac{\sinh\frac{n\pi}{b}(a-x)}{\sinh n\pi\frac{a}{b}}\int_0^x f(\xi)\sinh\frac{n\pi\xi}{b}d\xi - \frac{2}{b}\frac{\sinh\frac{n\pi x}{b}}{\sinh n\pi\frac{a}{b}}\int_x^a f(\xi)\sinh\frac{n\pi}{b}(a-\xi)d\xi \end{aligned} \tag{F.31}$$

Substitution into (F.13) gives the solution for $p(x,y)$ in terms of its boundary values $p(x,0) = f(x)$ on the side of the rectangular region on the x-axis:

$$p(x,y) = -\frac{2}{b}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi y}{b}}{\sin\frac{n\pi a}{b}}\left\{ \sinh\frac{n\pi}{b}(a-x)\int_0^x f(\xi)\sinh\frac{n\pi\xi}{b}d\xi + \sinh\frac{n\pi x}{b}\int_x^a f(\xi)\sinh\frac{n\pi}{b}(a-\xi)d\xi \right\} \tag{F.32}$$

This result gives $p(x,0) = 0$ on the side of the rectangle on the x-axis—where we have already specified, and presumably satisfied, the condition $p(x,0) = f(x)$. We therefore have the paradoxical situation quite analogous to that prevailing for the acoustic field treated with the method described in Chapter 4: The solution (F.32) has been constructed to satisfy the boundary condition (F.11) on $y = 0$, but because $\sin\frac{n\pi y}{b}$ is zero on $y = 0$, the specified boundary condition is obviously not satisfied.

Owing to the relative simplicity of the formulas arising in this problem, we can resolve the apparent paradox explicitly and quite easily. The key to doing so is to show that (F.32) is *not* a continuous function of y as $y \to 0$, expressed by the statements

(i) $p(x,0) = 0$, which follows immediately from (F.32); but

(ii) $\lim_{y\to\infty} p(x,y) = f(x)$, the required boundary condition (F.11)

[1] Because the differential operator defined by (F.15) is self-adjoint, the associated Green's is symmetric (Morse and Feshbach 1948, for example).

Another way of stating this result is: "the limit of the sum is not equal to the sum of the limits" $\int \sum \neq \sum \int$.

We already know by inspection that $\sum \int = 0$ for (F.32), the result following because $\sin \frac{n\pi y}{b} = 0$ for $y = 0$ in all terms of the finite sum, and then letting the sum become infinite. So we have only to show that statement (ii) follows from the solution (F.32). Write (F.32) in the form (F.13),

$$p(x, y) = \sum_{n=1}^{\infty} c_n(x) \sin \frac{n\pi y}{b}$$

where $c_n(x)$ is given by (F.31). Integrate by parts, the two terms in $c_n(x)$,

$$\int_0^x f(\xi) \sinh \frac{n\pi\xi}{b} d\xi = \frac{b}{n\pi} \left[f(\xi) \cosh \frac{n\pi\xi}{b} \right]_0^x - \frac{b}{n\pi} \int_0^x \frac{df}{d\xi} \cosh \frac{n\pi\xi}{b} d\xi$$

$$\int_x^a f(\xi) \sinh \frac{n\pi}{b}(a-\xi) d\xi = -\frac{b}{n\pi} \left[f(\xi) \cosh \frac{n\pi}{b}(a-\xi) \right]_x^a - \frac{b}{n\pi} \int_x^a \frac{df}{d\xi} \cosh \frac{n\pi}{b}(a-\xi) d\xi$$

The result of substituting into the formula (F.30) can be put in the form

$$c_n(x) = -\frac{2}{n\pi} f(x) + C_n(x) \tag{F.33}$$

where

$$\begin{aligned} C_n(x) = \frac{2}{n\pi} \frac{1}{\sinh \frac{n\pi a}{b}} \Bigg\{ & \left[f(a) \sinh \frac{n\pi x}{b} - f(0) \sinh \frac{n\pi}{b}(a-x) \right] \\ & + \left[\int_0^x \frac{df}{d\xi} \cosh \frac{n\pi\xi}{b} d\xi - \int_x^a \frac{df}{d\xi} \cosh \frac{n\pi}{b}(a-\xi) d\xi \right] \Bigg\} \end{aligned} \tag{F.34}$$

We take advantage of a result used by Friedman (1956, p. 271) and assumed here without proof. For the Fourier series

$$S = \sum_{n=1}^{\infty} A_n \sin ny,$$

suppose that for n large,

$$A_n = \frac{a_1}{n} + \frac{a_2(n)}{n^2}$$

where a, is constant and $a_2(n)$ is bounded. If $a_1 \neq 0$,

$$S = a_1 \sum_{n=1}^{\infty} \frac{\sin ny}{n} + \sum_{n=1}^{\infty} \frac{a_2(n)}{n^2} \sin ny$$

The second sum tends to zero as $y \to 0$ and the first sum is $a_1(\pi - y)/2$ for $0 < y < \pi$. Hence, for $y \to 0$,

$$S = \sum_{n=1}^{\infty} A_n \sin n\pi \underset{y\to\infty}{\longrightarrow} a_1 \frac{\pi - y}{2} + \sum_{n=1}^{\infty} \frac{a_2(n)}{n^2} y \underset{y=0}{=} a_1 \frac{\pi}{2} \tag{F.35}$$

Therefore, $p(x, y)$ given by (F.13) with $c_n(x)$ expressed as (F.33) is

$$p(x, y) = \frac{2}{\pi} f(x) \sum_{n=1}^{\infty} \frac{\sin ny}{n} + \sum_{n=1}^{\infty} C_n(x) \sin ny$$

For $y \to 0$,

$$p(x, y \to 0) = \frac{2}{\pi} f(x) \left(\frac{\pi - y}{2} \right) + \sum_{n=1}^{\infty} C_n(x) \sin ny$$

which for $y = 0$ becomes

$$p(x, 0) = f(x) \tag{F.36}$$

This then demonstrates that with sufficient care, the representation (F.13) which apparently cannot satisfy the boundary condition $p \neq 0$ on $y = 0$, in fact *does* if the infinite series is properly summed.

We have not, and will not here, prove the corresponding property for our case summarized by equations (F.1)–(F.7), but throughout this book, whenever necessary, we assume its truth.