ANNEX H

Some Alternative Methods of Approximately Analyzing Nonlinear Behavior

Time-dependent nonlinear behavior has long been an object of intensive research in several other fields, notably electrical and earthquake engineering. Some of the results obtained in this book, or cited, originated in quite different contexts. Probably the main distinguishing feature of the applications motivating much of the present work is the emphasis on the transient growth of a small amplitude motion into a limit cycle executed by a (largely) isolated physical system. Particularly for combustion instabilities in solid propellant rockets, much effort has been devoted to improving the precision of the results; and to broadening applicability of those results to complicated internal configurations. The purpose of this brief annex is solely to call attention to other methods which may be attractive for particular applications. No derivations or results are given and the citations are intended only to provide an entry into the literature. They are neither the most recent nor are they necessarily the most representative works.

Probably the practical methods most commonly encountered are known by the names *harmonic linearization*; *harmonic balance*; *equivalent linearization*; *quasi-linearization*; and *describing function analysis*. Perhaps the best, through brief, treatment for methods useful in the subject of this book is the discussion Natanzon (1999) gives in his Chapter Five. This is a particularly interesting brief coverage of some of the Russian work which evidently was developed for application to liquid rockets, significantly well in advance of comparable analyses in the West. The approaches rest ultimately on the influential work by Krylov and Bogoliubov (1943), Bogoliubov and Mitropolsky (1961) and Andronov, Vitt, and Khaikin (1966). A strong influence, particularly in the earlier works, is the behavior of electronic devices and the properties of radio waves. Attention to continuous systems and the associated systems of nonlinear partial differential equations was therefore not often a leading concern.

Many approximate methods for analyzing nonlinear behavior have developed from the assumption that the motions of the system at hand are dominated by oscillation at a single frequency. That is the essential basis for the first four of the methods listed at the beginning of the preceding paragraph. For example, Natanzon opens his survey with a development of *harmonic linearization*, partly a misnomer because it leads to a nonlinear governing equation which is then 'linearized'. Simplification of the nonlinear behavior is achieved partly by initially treating steady oscillations occurring in limit cycles. Slow variations are handled in a manner similar to that explained by Krylov and Bogoliubov. Natanzon then discusses "hard excitation" of oscillations, i.e., pulsed nonlinear instabilities, with a simple combination of analytical results and graphical interpretation. It is important to note that, because by assumption motion at one frequency suppresses other frequencies, there is no possibility of treating such phenomena as, for example, energy transfer between modes. However, consequences of nonlinear behavior means the assumed single harmonic can be treated in a heuristic fashion. Natazon (his Section 5.2) does so using a form of the n–τ model of combustion. It is an interesting peek at a piece of early Russian work on nonlinear combustion instabilities. See also MacDonald (1993) for a variation of the technique called the *harmonic balance method*.

Methods of *equivalent linearization* have been very popular in earthquake engineering (e.g., Caughey, 1963; Iwan 1973, 1978) but apparently originated in mechanics (Minorsky 1947; Bogoliubov and Mitropolski

1961). As the term 'equivalent linearization' suggests, the idea is to replace a nonlinear problem—typically involving the motion of an oscillator—by a linear form. Much of the theory which has been developed is directed to minimizing the error incurred in the process. Caughey (1963), for example, has discussed the method applied to nonlinear oscillators subject to random forces. Iwan (1978) has given a useful discussion of the connections between the methods of equivalent linearization and harmonic balance.

Especially in the field of automatic control—with or without feedback—the method based on *describing functions* has long been favored. It is also the method most easily learned, appearing at length in several texts, e.g., Truxal (1955); Graham and McRuer (1961); and Ogata (1970); and the quite thorough treatment of Gelb and Vander Velde (1968). The quickest way to obtain an extended introduction is to search on the web. You should not expect to find applications to combustion systems. But because the point of view developed in this book emphasizes the central role of oscillations, the discussions in the references will often expand one's understanding and perhaps suggest novel calculations.

Ogata (p. 652) defines the *describing function* as "the complex ratio of the fundamental harmonic component of the output to the input." Hence the describing function is an extension to nonlinear systems of the definition of the transfer function for linear systems. The earliest description of the method was by Kochenburger (1950) based on his MIT Ph.D. thesis. Truxal (1955) has briefly summarized background and has given a good introduction to the method. For the most extended treatment, see Gelb and Vander Velde (1968). One should keep these early references in context, for the field of dynamical systems, both theory and analysis, has of course advanced enormously in recent decades. Modern computing machinery has often had far-reaching qualitative as well as quantitative consequences, but the importance of approximate methods remains.