Randomized Algorithms for Systems and Control: Theory and Applications

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References

- R. Tempo, G. Calafiore and F. Dabbene, "Randomized Algorithms for Analysis and Control of Uncertain Systems," Springer-Verlag, London, 2005
- R. Tempo and H. Ishii, "Monte Carlo and Las Vegas Randomized Algorithms for Systems and Control: An Introduction," European Journal of Control, Vol. 13, pp. 189-203, 2007
- RACT: Randomized Algorithms Control Toolbox for Matlab

  http://ract.sourceforge.net

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Overview

- Preliminaries
- Randomized Algorithms for Analysis
- Probabilistic Robust Synthesis
- Randomized Algorithms for Optimal Control (LQR)
- Extensions
- Applications: Probabilistic Control of Mini UAVs

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Preliminaries

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Randomized Algorithms (RAs)

- Randomized algorithms are frequently used in many areas of engineering, computer science, physics, finance, optimization,…but their appearance in systems and control is mostly limited to Monte Carlo simulations…
- Main objective of this mini-course: Introduction to rigorous study of RAs for uncertain systems and control, with specific applications

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Randomized Algorithms (RAs)

- Combinatorial optimization, computational geometry
- Examples: Data structuring, search trees, graph algorithms, sorting (RQS), …
- Motion and path planning problems
- Mathematics of finance: Computation of path integrals
- Bioinformatics (string matching problems)

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Uncertainty

- Uncertainty has been always a critical issue in control theory and applications
- First methods to deal with uncertainty were based on a stochastic approach
- Optimal control: LQG and Kalman filter
- Since early 80's alternative deterministic approach (worst-case or robust) has been proposed

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Robustness

- Major stepping stone in 1981: Formulation of the $\mathcal{H}_\infty$ problem by George Zames
- Various "robust" methods to handle uncertainty now exist: Structured singular values, Kharitonov, optimization-based (LMI), *l*-one optimal control, quantitative feedback theory (QFT)

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Robustness

- Late 80's and early 90's: Robust control theory became a well-assessed area
- Successful industrial applications in aerospace, chemical, electrical, mechanical engineering, …
- However, …

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Limitations of Robust Control - 1

- Researchers realized some drawbacks of robust control
- Consider uncertainty Δ bounded in a set $\mathcal{B}$ of radius ρ. Largest value of ρ such that the system is stable for all $\Delta \in \mathcal{B}$ is called (worst-case) robustness margin
- Conservatism: Worst case robustness margin may be small
- Discontinuity: Worst case robustness margin may be discontinuous wrt problem data

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Limitations of Robust Control - 2

- Computational Complexity: Worst case robustness is often $\mathcal{NP}$-hard (not solvable in polynomial time unless $\mathcal{P} = \mathcal{NP}$)[1]
- Various robustness problems are $\mathcal{NP}$-hard
  - static output feedback
  - structured singular value
  - stability of interval matrices

[1] V. Blondel and J.N. Tsitsiklis (2000)

Conservatism and Complexity Trade-Off

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- Uncertain or control design parameters often enter into the system in a nonlinear/nonconvex fashion
- To avoid complexity issues (or just to find a solution of the problem) relaxation techniques such as SOS are used
- Study issues about the accuracy of the approximation introduced and related complexity

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Different Paradigm Proposed

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- New paradigm proposed is based on uncertainty randomization and leads to randomized algorithms for analysis and synthesis
- Within this setting a different notion of problem tractability is needed
- Objective: Breaking the curse of dimensionality[1]

[1] R. Bellman (1957)

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Probability *and* Robustness

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- The interplay of Probability and Robustness for control of uncertain systems
- Robustness: Deterministic uncertainty bounded
- Probability: Random uncertainty (pdf is known)
- Computation of the probability of performance
- Controller which stabilizes *most* uncertain systems

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Key Features

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- We obtain larger robustness margins at the expense of a small risk
- We study the probability degradation *beyond* the robustness margins
- Computational complexity is generally not an issue: Randomized algorithms are low complexity

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Uncertain Systems

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$M(s)$ System	Δ Uncertainty

- Δ belongs to a structured set $\mathcal{B}_D$
  - Parametric uncertainty q
  - Nonparametric uncertainty Δ_i
  - Mixed uncertainty

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Worst Case Model

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- Worst case model: Set membership uncertainty
- The uncertainty Δ is bounded in a set $\mathcal{B}_D$

$$\Delta \in \mathcal{B}_D$$

- Real parametric uncertainty $q=[q_1,\ldots, q_\ell] \in \mathbb{R}^\ell$

$$q_i \in [q_i^-, q_i^+]$$

- Nonparametric uncertainty

$$\Delta_i \in \{\Delta_i \in \mathbb{R}^{n,n} : \|\Delta_i\| \le 1\}$$

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Robustness

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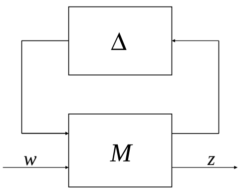

- Uncertainty Δ is bounded in a structured set $\mathcal{B}_D$
- $z = \mathcal{F}_u(M,\Delta)\, w$, where $\mathcal{F}_u(M,\Delta)$ is the upper LFT

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Objective of Robustness

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- Objective of robustness: To guarantee stability and performance for all

$$\Delta \in \mathcal{B}_D$$

- Different probabilistic paradigm based on uncertainty randomization of Δ within $\mathcal{B}_D$

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Example: Flexible Structure - 1

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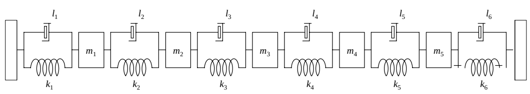
- Mass spring damper model
- Real parametric uncertainty affecting stiffness and damping
- Complex unmodeled dynamics (nonparametric)



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Flexible Structure - 2

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- M-Δ configuration for controlled systems and study stability of

$$M(s) = C(sI - A)^{-1}B$$

$$\Delta = \begin{bmatrix} q_1 I_5 & 0 & 0 \\ 0 & q_2 I_5 & 0 \\ 0 & 0 & \Delta_1 \end{bmatrix}$$

$q_1, q_2 \in \mathbb{R}$

$\Delta_1 \in \mathbb{C}^{4,4}$

$\Delta \in \mathcal{B}_D = \{\Delta \in \mathcal{D} : \sigma(\Delta) < \rho\}$

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Probability Degradation Function

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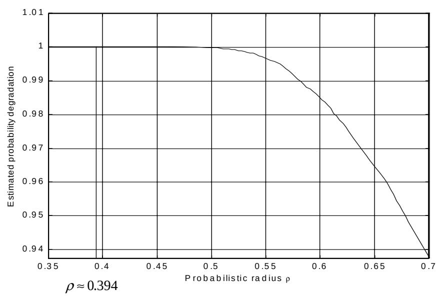

$\rho \approx 0.394$

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Probabilistic Model

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- Probability density function associated to $\mathcal{B}_D$
- We now assume that Δ is a random matrix with given density function $f_\Delta(\Delta)$ and support $\mathcal{B}_D$
- Example: Δ is uniform in $\mathcal{B}_D$

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Uniform Density

- Take $f_\Delta(\Delta)=\mathcal{U}[\mathcal{B}_D]$ (uniform density within $\mathcal{B}_D$)

$$\mathcal{U}[\mathcal{B}_D] = \begin{cases} \dfrac{1}{vol(\mathcal{B}_D)} & if\ \Delta \in \mathcal{B}_D \\ 0 & otherwise \end{cases}$$

- In this case, for a subset $\mathcal{S} \subseteq \mathcal{B}_D$

$$\Pr\{\Delta \in \mathcal{S}\} = \frac{\int_{\mathcal{S}} d\Delta}{vol(\mathcal{B}_D)} = \frac{vol(\mathcal{S})}{vol(\mathcal{B}_D)}$$

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Performance Function

- In classical robustness we guarantee that a certain performance requirement is attained for all $\Delta \in \mathcal{B}_D$
- This can be stated in terms of a performance function

$$J = J(\Delta)$$

- Examples: $\mathcal{H}_\infty$ performance and robust stability

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Example: $\mathcal{H}_\infty$ Performance - 1

- Compute the $\mathcal{H}_\infty$ norm of the upper LFT $\mathcal{F}_u(M,\Delta)$

$$J(\Delta) = \| \mathcal{F}_u(M, \Delta)\|_\infty$$

- For given $\gamma>0$, check if

$$J(\Delta) < \gamma$$

for all Δ in $\mathcal{B}_D$

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Example: $\mathcal{H}_\infty$ Performance - 2

- Continuous time SISO systems with real parametric uncertainty q with upper LFT

$$\mathcal{F}_u(M,\Delta) = \mathcal{F}_u(M,q) =$$

$$\frac{0.5q_1q_2s + 10^{-5}q_1}{(10^{-5} + 0.05q_2)s^2 + (0.00102 + 0.5q_2)s + (2\cdot 10^{-5} + 0.5q_1^2)}$$

where $q_1 \in [0.2, 0.6]$ and $q_2 \in [10^{-5}, 3\cdot 10^{-5}]$

- Letting $J(q) = \| \mathcal{F}_u(M,q)\|_\infty$, we choose γ=0.003
- Check if $J(q)<\gamma$ for all q in these intervals

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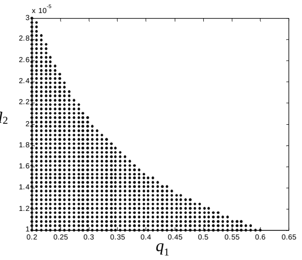
Example: $\mathcal{H}_\infty$ Performance - 3

- The set of q_1, q_2 for which $J(q)<\gamma$ is shown below



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Example[1]: Robust Stability - 1

- Consider the closed loop uncertain polynomial

$$p(s,q) =$$

$$(1+r^2+6q_1+6q_2+2q_1q_2)+(q_1+q_2+3)s+(q_1+q_2+1)s^2+s^3$$

where $q_1 \in [0.3, 2.5]$, $q_2 \in [0,1.7]$ and r=0.5

- Check stability for all q in these intervals

[1] G. Truxal (1961)

Example: Robust Stability - 2

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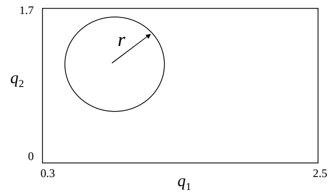
- Set of unstable polynomials



- Taking r=0 the unstable set reduces to a singleton

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Good and Bad Sets

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- We define two subsets of $\mathcal{B}_D$

$$\Delta_{good} = \{\Delta: J(\Delta) \leq \gamma\} \subseteq \mathcal{B}_D$$
$$\Delta_{bad} = \{\Delta: J(\Delta) > \gamma\} \subseteq \mathcal{B}_D$$

- Δ_{good} is the set of Δ's satisfying performance
- Measure of robustness is

$$vol(\Delta_{good}) = \int_{\Delta_{good}} d\Delta$$

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Example of Good and Bad Sets

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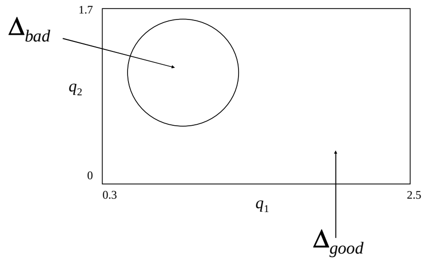

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Example of Good and Bad Sets - 2

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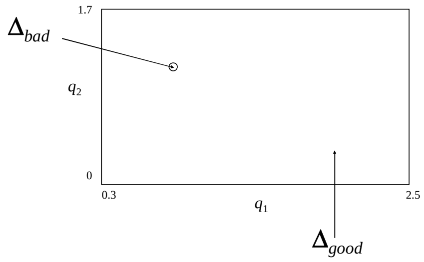
Taking small r



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Probabilistic Robustness Measure

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- In worst-case analysis we compute γ such that all Δ satisfy performance. Equivalently, we evaluate γ such that

$$\Delta_{good} = \mathcal{B}_D$$

- In a probabilistic setting, we are satisfied if the ratio

$$\frac{vol(\Delta_{good})}{vol(\mathcal{B}_D)}$$

is close to one

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Probability of Performance[1]

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- We define the probability of performance as

$$p_\gamma = \Pr\{J(\Delta) \leq \gamma\}$$

- Notice that, if $f_\Delta(\Delta)$ is uniform, then

$$p_\gamma = \frac{vol(\Delta_{good})}{vol(\mathcal{B}_D)}$$

[1] R.F. Stengel (1980)

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Example: Closed-Form Computation

- For Truxal's example, we compute p_γ in closed-form
- For uniform distribution, we have

$$vol(\Delta_{good}) = 3.74 - \pi\, r^2$$

$$vol(\mathcal{B}_D) = 3.74$$

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P1: Performance Verification

- For given performance level γ, check whether

$$J(\Delta) \le \gamma$$

for all Δ in $\mathcal{B}_D$

- Compute the probability of performance p_γ

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P2: Worst-Case Performance

- Find $J_{\max}$ such that

$$J_{\max} = \max_{\Delta \in \mathcal{B}_D} J(\Delta)$$

- Compute the worst case performance (or its probabilistic counterpart)

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Randomized Algorithms for Analysis

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Randomized Algorithm: Definition

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result
- Example of a "random choice" is a coin toss

heads or *tails*

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Randomized Algorithm: Definition

- Randomized Algorithm (RA): An algorithm that makes random choices during its execution to produce a result
- For hybrid systems, "random choices" could be switching between different states or logical operations
- For uncertain systems, "random choices" require (vector or matrix) random sample generation

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Monte Carlo Randomized Algorithm

- Monte Carlo Randomized Algorithm (MCRA): A randomized algorithm that may produce incorrect results, but with bounded error probability

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Las Vegas Randomized Algorithm

- Las Vegas Randomized Algorithm (LVRA): A randomized algorithm that always produces correct results, the only variation from one run to another is the running time

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Randomization of Uncertain Systems

- Consider random uncertainty Δ, associated pdf and bounding set B
- Δ is a (real or complex) random vector (parametric uncertainty) or matrix (nonparametric uncertainty)
- Consider a performance function

$$J(\Delta): B \rightarrow \mathbf{R}$$

and level $\gamma > 0$

- Define worst case and average performance

$$J_{\max} = \max_{\Delta \in B} J(\Delta) \qquad J_{\text{ave}} = E_{\Delta}(J(\Delta))$$

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Example: H_∞ Performance - 1

- H_∞ performance of sensitivity function

$$B = \{\Delta \colon \Delta = \text{bdiag}\,(\Delta_1, \ldots, \Delta_q) \in \mathbf{F}^{n,m},\ \sigma_{max}(\Delta) \le \rho\}$$

$$S(s,\Delta) = 1/(1 + P(s,\Delta)\, C(s))$$

$$J(\Delta) = \|S(s,\Delta)\|_\infty$$

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Example: H_∞ Performance - 2

- H_∞ performance of sensitivity function

$$B = \{\Delta \colon \Delta = \text{bdiag}\,(\Delta_1, \ldots, \Delta_q) \in \mathbf{F}^{n,m},\ \sigma_{\max}(\Delta) \le \rho\}$$

$$S(s,\Delta) = 1/(1 + P(s,\Delta)\, C(s))$$

$$J(\Delta) = \|S(s,\Delta)\|_\infty$$

- Objective: Check if

$$J_{\max} \leqslant \gamma \quad \text{and} \quad J_{\text{ave}} \leqslant \gamma$$

- These are uncertain decision problems

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Two Problem Instances

- We have two problem instances for worst case performance

$$J_{\max} \leqslant \gamma \quad \text{and} \quad J_{\max} > \gamma$$

and two problem instances for average case performance

$$J_{\text{ave}} \leqslant \gamma \quad \text{and} \quad J_{\text{ave}} > \gamma$$

- This leads to one-sided and two-sided MCRA

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One-Sided MCRA

- One-sided MCRA: Always provide a correct solution in one of the instances (they may provide a wrong solution in the other instance)
- Consider the empirical maximum

$$\hat{J}_{max} = \max_{i=1,\dots,N} J(\Delta^i)$$

where Δ^i are random samples and N is the sample size

- Check if $\hat{J}_{max} \leqslant \gamma$ or $\hat{J}_{max} > \gamma$

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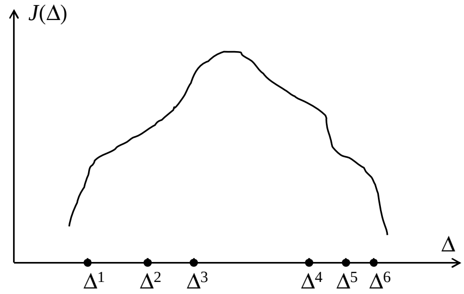
One-Sided MCRA: Case 1



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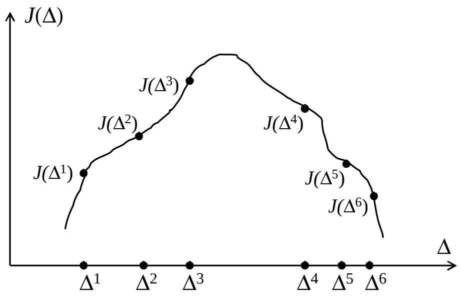
One-Sided MCRA: Case 1



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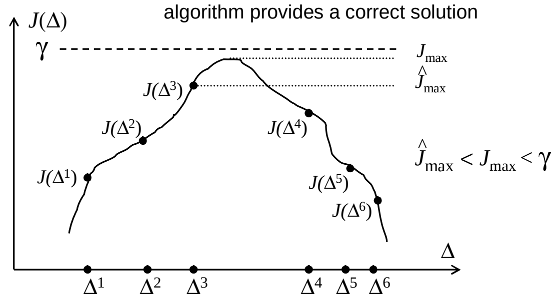
One-Sided MCRA: Case 1

algorithm provides a correct solution



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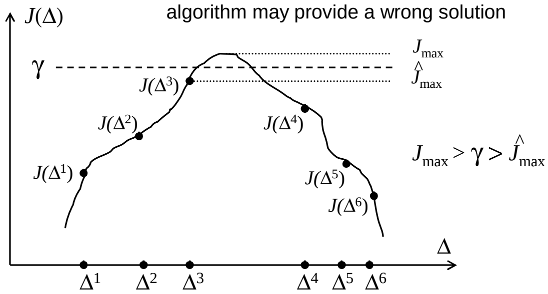
One-Sided MCRA: Case 2

algorithm may provide a wrong solution



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Two-Sided MCRA

- Two-sided MCRA: They may provide a wrong solution in both instances
- Consider the empirical average

$$\hat{J}_{ave} = \operatorname*{ave}_{i=1,\dots,N} J(\Delta^i)$$

where Δ^i are random samples and N is the sample size

- Check if $\hat{J}_{ave} \leqslant \gamma$ or $\hat{J}_{ave} > \gamma$

Two-Sided MCRA: Case 1

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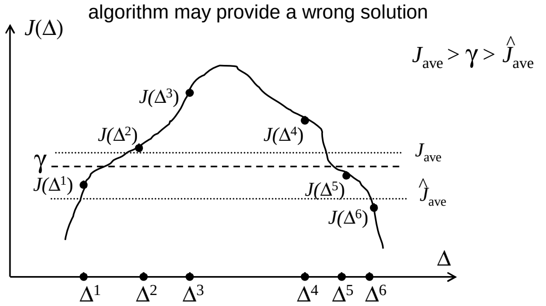
algorithm may provide a wrong solution



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Two-Sided MCRA: Case 2

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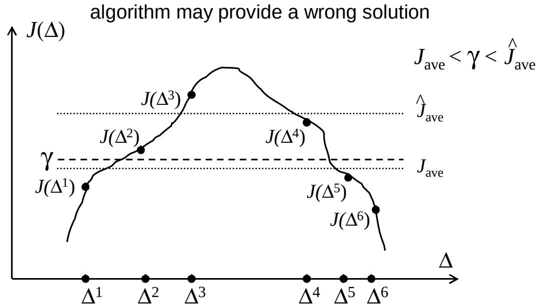
algorithm may provide a wrong solution



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Las Vegas Randomized Algorithms

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- We also have zero-sided (Las Vegas) randomized algorithms
- Las Vegas Randomized Algorithm (LVRA): Always give the correct solution
- The solution obtained with a LVRA is probabilistic, so “always” means with probability one
- Running time may be different from one run to another
- We can study the *average* running time

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Discrete Bounding Set

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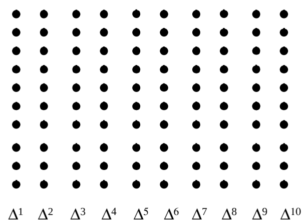
Consider random uncertainty Δ, a *discrete* bounding set $\mathcal{B}$, given pdf and performance function



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The Las Vegas Viewpoint

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- Consider discrete random variables
- The sample space is discrete and M^N possible choices can be made
- In the binary case we have 2^N
- Finding maximum requires ordering the 2^N choices
- Las Vegas can be used for ordering real numbers
- Example: Randomized Quick Sort for sorting real numbers (classical in computer science)

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Ingredients for RAs

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- Assume that Δ is random with pdf $f_\Delta(\Delta)$ with support $\mathcal{B}_D$
- Accuracy $\varepsilon \in (0,1)$ and confidence $\delta \in (0,1)$ be assigned
- Performance function for analysis and level

$$\downarrow \qquad\qquad \downarrow$$

$$J = J(\Delta) \qquad\qquad \gamma$$

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Randomized Algorithms for Analysis

- Two classes of randomized algorithms for probabilistic robust performance analysis

- P1: Performance verification (compute p_γ)
- P2: Worst-case performance (compute $J_{\max}$)

- Both are based on uncertainty randomization of Δ
- Bounds on the sample size are obtained

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Randomized Algorithms - 2

- We estimate p_γ by means of a randomized algorithm
- First, we generate N i.i.d. samples

$$\Delta^1, \Delta^2, \ldots, \Delta^N \in \mathcal{B}_D$$

according to the density f_Δ

- We evaluate $J(\Delta^1), J(\Delta^2), \ldots, J(\Delta^N)$

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Empirical Probability

- Construct an indicator function

$$I(\Delta^i) = \begin{cases} 1 & \textit{if } J(\Delta^i) \le \gamma \\ 0 & \textit{otherwise} \end{cases}$$

- An estimate of p_γ is the empirical probability

$$\hat{p}_N = \frac{1}{N}\sum_{i=1}^{N} I(\Delta^i) = \frac{N_{good}}{N}$$

where N_{good} is the number of samples such that $J(\Delta^i) \le \gamma$

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A Reliable Estimate

- The empirical probability is a reliable estimate if

$$|p_\gamma - \hat{p}_N| = |\Pr\{J(\Delta) \le \gamma\} - \hat{p}_N| \le \varepsilon$$

- Find the minimum N such that

$$\Pr\{|p_\gamma - \hat{p}_N| \le \varepsilon\} \ge 1 - \delta$$

where $\varepsilon \in (0,1)$ and $\delta \in (0,1)$

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Chernoff Bound[1]

- For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, if

$$N \ge \frac{\log \frac{2}{\delta}}{2\varepsilon^2}$$

then $$\Pr\{|p_\gamma - \hat{p}_N| \le \varepsilon\} \ge 1 - \delta$$

[1] H. Chernoff (1952)

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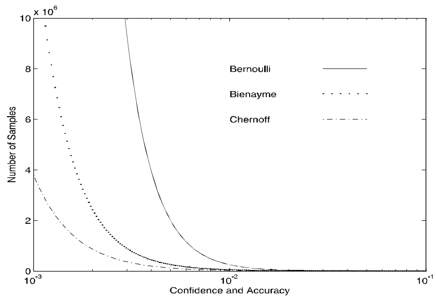
Comparison Between Bounds



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Chernoff Bound

- Remark: Chernoff bound improves upon other bounds such as Bernoulli (Law of Large Numbers)
- Dependence on $1/\delta$ is logarithmic
- Dependence on $1/\varepsilon$ is quadratic

ε	0.1%	0.1%	0.5%	0.5%
1-δ	99.9%	99.5%	99.9%	99.5%
N	$3.9 \cdot 10^6$	$3.0 \cdot 10^6$	$1.6 \cdot 10^6$	$1.2 \cdot 10^5$

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Computational Complexity of RAs

- RAs are efficient (polynomial-time) because

1. Random sample generation of Δ^i can be performed in polynomial-time
2. Cost associated with the evaluation of $J(\Delta^i)$ for fixed Δ^i is polynomial-time
3. Sample size is polynomial in the problem size and probabilistic levels ε and δ

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1. Random Sample Generation

- Random number generation (RNG): Linear and nonlinear methods for uniform generation in [0,1) such as Fibonacci, feedback shift register, BBS, MT, …
- Non-uniform univariate random variables: Suitable functional transformations (e.g., the inversion method)
- The problem is much harder: Multivariate generation of samples of Δ with pdf $f_\Delta(\Delta)$ and support $\mathcal{B}_D$
- It can be resolved in polynomial-time

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2. Cost of Checking Stability

- Consider a polynomial

$$p(s,a) = a_0 + a_1 s + \cdots + a_n s^n$$

- To check left half plane stability we can use the Routh test. The number of multiplications needed is

$$\frac{n^2}{4} \text{ for } n \text{ even} \qquad \frac{n^2-1}{4} \text{ for } n \text{ odd}$$

- The number of divisions and additions is equal to this number
- We conclude that checking stability is $O(n^2)$

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3. Bounds on the Sample Size

- Chernoff bound is independent on the size of $\mathcal{B}_D$, on the structure $\mathcal{D}$ on the number of blocks, on the pdf $f_\Delta(\Delta)$
- It depends only on δ and ε
- Same comments can be made for other bounds such as Bernoulli

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Worst-Case Performance

- Recall that

$$J_{\max} = \max_{\Delta \in \mathcal{B}_D} J(\Delta)$$

- Generate N i.i.d. samples

$$\Delta^1, \Delta^2, \ldots, \Delta^N \in \mathcal{B}_D$$

according to the density f_Δ

- Compute the empirical maximum

$$\hat{J}_{\max} = \max_{i=1,\ldots,N} J(\Delta^i)$$

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Worst-Case Bound (Log-over-Log)[1]

- For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, if

$$N \geq \frac{\log \frac{1}{\delta}}{\log \frac{1}{1-\varepsilon}}$$

then

$$\Pr\left\{\Pr\left\{J(\Delta) > \hat{J}_N\right\} \leq \varepsilon\right\} \geq 1-\delta$$

[1] R. Tempo, E. W. Bai and F. Dabbene (1996)

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Comparison and Comments

- Number of samples is much smaller than Chernoff
- Bound is a specific instance of the fpras (fully polynomial randomized approximated scheme) theory
- Dependence on $1/\varepsilon$ is basically linear $\left(\log \frac{1}{1-\varepsilon} \approx \varepsilon\right)$

ε	0.1%	0.1%	0.5%	0.5%	0.01%	0.001%
1-δ	99.9%	99.5%	99.9%	99.5%	99.99%	99.999%
N	$6.91 \cdot 10^3$	$5.30 \cdot 10^3$	$1.38 \cdot 10^3$	$1.06 \cdot 10^3$	$9.21 \cdot 10^4$	$1.16 \cdot 10^6$

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Volumetric Interpretation

- In the case of $f_\Delta(\Delta)$ uniform, we have

$$\Pr\left\{J(\Delta) > \hat{J}_N\right\} = \frac{vol(\Delta_{bad})}{vol(\mathcal{B}_D)}$$

- Therefore

$$\Pr\left\{\Pr\left\{J(\Delta) > \hat{J}_N\right\} \leq \varepsilon\right\} \geq 1-\delta$$

is equivalent to

$$\Pr\left\{vol(\Delta_{bad}) \leq \varepsilon\, vol(\mathcal{B}_D)\right\} \geq 1-\delta$$

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Confidence Intervals

- The Chernoff and worst-case bounds can be computed *a-priori* and provide an explicit functional relation

$$N = N(\varepsilon, \delta)$$

- The sample size obtained with the confidence intervals is not explicit
- Given $\delta \in (0,1)$, upper and lower confidence intervals p_L and p_U are such that

$$\Pr\{p_L \leq p_\gamma \leq p_U\} = 1-\delta$$

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Confidence Intervals - 2

- The probabilities p_L and p_U can be computed *a posteriori* when the value of N_{good} is known, solving equations of the type

$$\sum_{k=N_{good}}^{N} \binom{N}{k} p_L^k (1-p_L)^{N-k} = \delta_L$$

$$\sum_{k=0}^{N_{good}} \binom{N}{k} p_U^k (1-p_U)^{N-k} = \delta_U$$

with $\delta_L + \delta_U = \delta$

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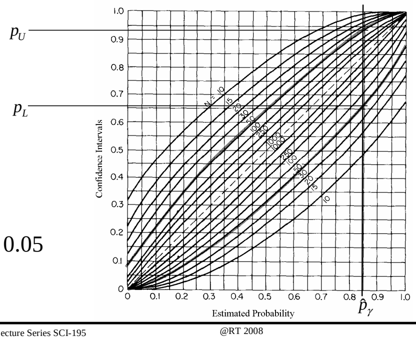
Confidence Intervals - 3



$\delta = 0.05$

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Statistical Learning Theory

- The Chernoff Bound studies the problem

$$\Pr\{|p_\gamma - \hat{p}_N| \le \varepsilon\} \ge 1-\delta$$

where $p_\gamma = \Pr\{J(\Delta) \le \gamma\}$

- Performance function J is fixed
- Statistical Learning Theory computes bounds on the sample size for the problem

$$\Pr\{|\Pr(J(\Delta) \le \gamma) - \hat{p}_N| \le \varepsilon, \forall J \in \mathcal{J}\} \ge 1-\delta$$

where $\mathcal{J}$ is a given class of functions

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VC and P-dimension[1,2]

- Statistical Learning Theory aims at studying uniform Law of Large Numbers
- The bounds obtained depend on quantities called VC-dimension (if J is a binary valued function), or P-dimension (if J is a continuous valued function)
- VC and P-dimension are measures of the problem complexity

[1] M. Vidyasagar (1997)
[2] E.D. Sontag (1998)

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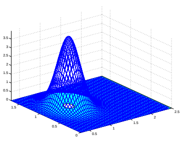
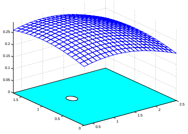
Choice of the Distribution - 1

- The probability $\Pr\{\Delta \in \mathcal{S}\}$ depends on $f_\Delta(\Delta)$
- It may vary between 0 and 1 depending on the pdf $f_\Delta(\Delta)$





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Choice of the Distribution - 2

- The bounds discussed are independent on the choice of the distribution but for computing $\Pr\{J(\Delta) \le \gamma\}$ we need to know the distribution $f_\Delta(\Delta)$
- Some research has been done in order to find the worst-case distribution in a certain class[1]
- Uniform distribution is the worst-case if a certain target is convex and centrally symmetric

[1] B. R. Barmish and C. M. Lagoa (1997)

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Choice of the Distribution - 3

- Minimax properties of the uniform distribution have been studied[1]

[1] E. W. Bai, R. Tempo and M. Fu (1998)

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Probabilistic Robust Synthesis

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Analysis vs Design with Uncertainty

- Starting point: Worst-case analysis versus design
- Consider an interval family $p(s,q)$, $q \in \mathcal{B}_q = \{q \in \mathbb{R}^n, \|q\|_\infty \leq 1\}$
- Analysis problem:
  - Check if $p(s,q)$ is stable for all $q \in \mathcal{B}_q$

    Answer: Kharitonov Theorem
- Design Problem:
  - Does there exist a $q \in \mathcal{B}_q$ such that $p(s,q)$ is stable?

    Answer: *Unknown* in general

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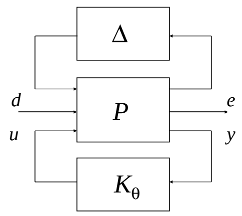
Synthesis Paradigm



- Design the parameterized controller K_θ to guarantee stability and performance

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Synthesis Performance Function

- Recall that the parameterized controller is K_θ
- We replace $J(\Delta)$ with a synthesis performance function

$$J = J(\Delta,\theta)$$

where $\theta \in \Theta$ represents the controller parameters to be determined and their bounding set

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Randomized Algorithms for Synthesis

- Two classes of RAs for probabilistic synthesis
- Average performance synthesis[1]
- Based on expected value minimization
- Use of Statistical Learning Theory results
- Very general problems can be handled
- Existing bounds are very conservative and controller randomization is required
- Ongoing research aiming at major reduction of sample size

[1] M. Vidyasagar (1998)

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Randomized Algorithms for Synthesis

- Robust performance synthesis[1]
- Problem reformulation as robust feasibility
- Only convex problems can be handled
- Finite-time convergence with probability one is obtained

[1]B. Polyak and R. Tempo (2001)

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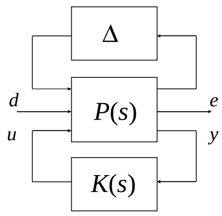
Robust Performance Synthesis

- Uncertainty randomization of Δ in $\mathcal{B}_D$
- Convex optimization to design the controller $K(s)$



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RAs for Optimal Control (LQR)

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Uncertain Systems in State Space

- We consider a state space description of the uncertain system

$$\dot{x}(t) = A(\Delta)x(t) + Bu(t)$$

with $x(0)=x_0$; $x\in\mathbb{R}^n$; $u\in\mathbb{R}^m$, $\Delta\in \mathcal{B}_D$

- For example, $A(\Delta)$ is an interval matrix with bounded entries $a_{ij}^- \le a_{ij} \le a_{ij}^+$

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Interval and Vertex Matrices

- We consider interval uncertainty A (i.e. when $\Delta\in \mathcal{B}_D$)
- That is, a_{ik} ranges in the interval for all i, k

$$|a_{ik} - a_{ik}^*| \le w_{ik}$$

where a_{ik}^* are nominal values and w_{ik} are weights

- Define the $N = 2^{n2}$ vertex matrices $A^1, A^2,\ldots, A^N$

$$a_{ik} = a_{ik}^* + w_{ik} \quad \text{or} \quad a_{ik} = a_{ik}^* - w_{ik}$$

for all i, k =1, 2, …, n

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Common Lyapunov Functions

- Given matrices A^*, W and feedback K, find a *common quadratic Lyapunov* function $Q > 0$ for the system

$$\dot{x}(t) = (A + B K)\, x(t) \quad \text{for all} \quad A\in A$$

- Find $Q > 0$ such that

$$L(Q, A) = (A+BK)^T Q + Q\,(A+BK) < 0 \quad \text{for all} \quad A \in A$$

- Equivalently, find $Q > 0$ such that

$$\lambda_{max}\, L(Q, A) < 0 \quad \text{for all} \quad A \in A$$

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Lyapunov Stability of Interval Systems

- Quadratic Lyapunov stability analysis and synthesis of interval systems are NP-hard problems
- In principle, they can be solved in one-shot with convex optimization, but the number of constraints is exponential
- We can use relaxation (e.g. $\pi/2$ Theorem[1]) or randomization

[1] Yu. Nesterov (1997)

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Vertex Solution

- Due to convexity, it suffices to study $L(Q, A) < 0$ for all vertex matrices[1]
- Question: Do we really need to check all the vertex matrices ($N = 2^{n2}$)?

[1] H.P. Horisberger, P.R. Belanger (1976)

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Vertex Reduction

- Answer: It suffices to check "only" a subset of 2^{2n} vertex matrices[1]
- This is still exponential (the problem is NP-hard), but it leads to a major computational improvement for medium size problems (e.g. n = 8 or 10)
- For example, for n=8, N is of the order 10^5 (instead of 10^{19})

[1] T. Alamo, R. Tempo, D. Rodriguez, E.F. Camacho (2007)

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Diagonal Matrices and Generalizations

- Transform the original problem from full square matrices A to diagonal matrices $Z \in \mathbf{R}^{2n,2n}$
- It suffices to check the vertices of Z
- Extensions for L_2-gain minimization and other related LMI problems
- Generalizations for multiaffine interval systems

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Las Vegas Randomized Algorithm

- We may perform randomization of the $N = 2^{n^2}$ vertices (in the worst case)
- If we select the vertices in random order according to a given pdf, we have a LVRA

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Probabilistic Solution

- Randomly generate $A^1,\ldots, A^N$. Then, check if the Lyapunov equation

$$A^iQ + Q(A^i)^T \leq 0$$

is feasible for i=1,…,N and find a common solution $Q=Q^T>0$

- Critical problem: Even if N is relatively small, this is a hard computational problem

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Sequential Algorithm

- Key point: Sequential algorithm which deals with one constraint at each step
- At step k we have

  Phase 1: Uncertainty randomization of Δ

  Phase 2: Gradient algorithm and projection
- Final result: Find a solution $Q=Q^T>0$ with probability one in a finite number of steps

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Definition

- Let $\mathcal{E}_n$ be an Euclidean space

$$\mathcal{E}_n = \left\{ A = A^T \in \mathbb{R}^n, \|A\| = \sqrt{\sum_{i,k=1}^{n} a_{1k}^2} \right\}$$

and C be the cone of positive semi-definite matrices

$$C = \{A \in \mathcal{E}_n : A \geq 0\}$$

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Projection on a Cone

- For any real symmetric matrix A we define the projection $[A]^{+} \in C$ as

$$[A]^{+} = \arg\min_{X \in C} \|A - X\|$$

- The projection can be computed through the eigenvalue decomposition $A = T \Lambda T^{T}$
- Then

$$[A]^{+} = T \Lambda^{+} T^{T}$$

where $\lambda_i^{+} = \max\{\lambda_i, 0\}$

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Phase 1: Uncertainty Randomization

- Uncertainty randomization: Generate $\Delta^{k} \in \mathcal{B}_D$
- Then, for guaranteed cost we obtain the Lyapunov equation

$$A(\Delta^{k})Q + QA^{T}(\Delta^{k}) \leq 0$$

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Matrix Valued Function

- Define a matrix valued function

$$V(Q, \Delta^{k}) = A(\Delta^{k})Q + QA^{T}(\Delta^{k})$$

and a scalar function

$$v(Q, \Delta^{k}) = \left\| \left[V(Q, \Delta^{k}) \right]^{+} \right\|$$

where $\|\cdot\|$ is the Frobenius norm
- We can also take the maximum eigenvalue of $V(Q, \Delta^{k})$

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Phase 2: Gradient Algorithm

- We write

$$Q^{k+1} = \begin{cases} \left[Q^{k} - \mu^{k} \partial_{Q} \left\{ v\left(Q^{k}, \Delta^{k}\right) \right\} \right]^{+} & \text{if } v\left(Q^{k}, \Delta^{k}\right) > 0 \\ Q^{k} & \text{otherwise} \end{cases}$$

where ∂_Q is the subgradient and the stepsize μ^{k} is

$$\mu^{k} = \frac{v\left(Q^{k}, \Delta^{k}\right) + r \left\| \partial_{Q} \left\{ v\left(Q^{k}, \Delta^{k}\right) \right\} \right\|}{\left\| \partial_{Q} \left\{ v\left(Q^{k}, \Delta^{k}\right) \right\} \right\|^{2}}$$

and r>0 is a parameter

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Closed-form Gradient Computation

- The function $v(Q, \Delta^{k})$ is convex in Q and its subgradient can be easily computed in a closed form

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Theorem[1]

- Assumption: Every open subset of $\mathcal{B}_D$ has positive measure
- Theorem: A solution Q, if it exists, is found in a finite number of steps with probability one
- Idea of proof: The distance of Q^{k} from the solution set decreases at each correction step

[1] B.T. Polyak and R. Tempo (2001)

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Example[1]

- We study a multivariable example for the design of a controller for the lateral motion of an aircraft.
- The model consists of four states and two inputs

$$\dot{x}(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_p & L_\beta & L_r \\ g/V & 0 & Y_\beta & -1 \\ N_{\dot\beta}(g/V) & N_p & N_\beta + N_{\dot\beta}Y_\beta & N_r - N_{\dot\beta} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t)$$

[1] B.D.O. Anderson and J.B. Moore (1971)

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Example - 2

- The state variables are
  - x_1 bank angle
  - x_2 derivative of bank angle
  - x_3 sideslip angle
  - x_4 jaw rate
- The control inputs are
  - u_1 rudder deflection
  - u_2 aileron deflection

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Example - 3

- Nominal values: L_p=-2.93, L_β=-4.75, L_r=0.78, g/V=0.086, Y_β=-0.11, $N_{\dot\beta}$=0.1, N_p=-0.042, N_β=2.601, N_r=-0.29
- Perturbed matrix $A(\Delta)$: each parameter can take values in a range of ±15% of the nominal value
- Quadratic stability (γ=0): take $R=I$ and $S=0.01I$
- Remark: $A(\Delta)$ is multiaffine in the uncertain parameters: quadratic stability can be ascertained solving simultaneously 2^9=512 LMIs

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Example - 4

- Sequential algorithm:
  - Initial point Q_0 randomly selected
  - 800 random matrices Δ^k
  - The algorithm converged to

$$Q=\begin{bmatrix} 0.7560 & -0.0843 & 0.1645 & 0.7338 \\ -0.0843 & 1.0927 & 0.7020 & 0.4452 \\ 0.1645 & 0.7020 & 0.7798 & 0.7382 \\ 0.7338 & 0.4452 & 0.7382 & 1.2162 \end{bmatrix}$$

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Example - 5

- The corresponding controller

$$K=B^TQ^{-1}=\begin{bmatrix} 38.6191 & -4.3731 & 43.1284 & -49.9587 \\ -2.8814 & -10.1758 & 10.2370 & -0.4954 \end{bmatrix}$$

  satisfies all the 512 vertex LMIs and therefore it is also a quadratic stabilizing controller in a deterministic sense
- The optimal LQ controller computed on the nominal plant satisfies only 240 vertex LMIs

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Extensions

Related Literature and Extensions

- Minimization of a measure of violation for problems that are not strictly feasible[1]
- Uncertainty in the control matrix, $B=B(\Delta)$, $\Delta\in \mathcal{B}_D$

  We take the feedback law

  $$u = YQ^{-1}x$$

  where Y and $Q=Q^T>0$ are design variables

[1] B.R. Barmish and P. Shcherbakov (1999)

Related Literature

- Related literature on optimization and adaptive control with linear constraints[1,2,3,4]
- Stochastic approximation algorithms have been widely studied in the stochastic control and optimization literature[6,7]

[1] S. Agmon (1954)
[2] T.S. Motzkin and I.J. Schoenberg (1954)
[3] B.T. Polyak (1964)
[4] V.A. Bondarko and V.A. Yakubovich (1992)

[6] H.J. Kushner and G.G. Yin (2003)
[7] J.C. Spall (2003)

Subsequent Research

- Design of common Lyapunov functions for switched systems[1]
- From common to piecewise Lyapunov functions[2]
- Ellipsoidal algorithm instead of gradient algorithm[3]
- Stopping rule which provides the number of steps[4]
- Other algorithms have been recently proposed[5-6]

[1] D. Liberzon and R. Tempo (2004)
[2] H. Ishii, T. Basar and R. Tempo (2005)
[3] S. Kanev, B. De Schutter and M. Verhaegen (2002)
[4] Y. Oishi and H. Kimura (2003)
[5] Y. Fujisaki and Y. Oishi (2007)
[6] T. Alamo, R. Tempo, D. R. Ramirez and E. F. Camacho (2007)

Optimization Problems[1]

- Extensions to optimization problems
- Consider convex function $f(x)$ and function $g(x,\Delta)$ convex in x for fixed Δ
- Semi-infinite (nonlinear) programming problem

  $$\min f(x)$$
  $$g(x,\Delta) \leq 0 \text{ for all } \Delta\in \mathcal{B}$$

- Reformulation as stochastic optimization
- Drawback: Convergence results are only asymptotic

[1] V. B. Tadic, S. P. Meyn and R. Tempo (2003)

Scenario Approach

- The scenario approach for convex problems[1]
- Non-sequential method which provides a one-shot solution for general convex problems
- Randomization of $\Delta\in \mathcal{B}$ and solution of a single convex optimization problem
- Derivation of a bound on the sample size[1]
- A new improved bound based on a pack-based strategy[2]

[1] G. Calafiore and M. Campi (2004)
[2] T. Alamo, R. Tempo and E.F. Camacho (2007)

Convex Semi-Infinite Optimization

- The semi-infinite optimization problem is

  $$\min c^T\theta \quad \text{subject to } f(\theta,\Delta)\leq 0 \quad \text{for all } \Delta\in\mathcal{B}$$

  where $f(\theta,\Delta)\leq 0$ is convex in θ for all $\Delta\in\mathcal{B}$
- We assume that this problem is either unfeasible or, if feasible, it attains a unique solution for all $\Delta\in\mathcal{B}$ (this assumption is technical and may be removed)
- We assume that $\theta\in\Theta\subseteq\mathbf{R}^n$

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Scenario Problem

- Using randomization, we construct a scenario problem
- Taking random samples Δ^i, $i = 1, 2, \ldots, N$, we construct

$$f(\theta, \Delta^i) \leq 0, \quad i = 1, 2, \ldots, N$$

and

$$\min c^T \theta \quad \text{subject to } f(\theta, \Delta^i) \leq 0, \quad i = 1, 2, \ldots, N$$

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Theorem[1]

- Theorem: For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, if

$$N \geq \left\lceil 2/\varepsilon \log(1/\delta) + 2n + 2n/\varepsilon \log(2/\varepsilon) \right\rceil$$

then, with probability no smaller than 1- δ

- either the scenario problem is unfeasible and then also the semi-infinite optimization problem is unfeasible
- or, the scenario problem is feasible, then its optimal solution $\hat{\theta}_N$ satisfies

$$\Pr\{ \Delta \in \mathcal{B} : f(\theta, \Delta) > 0 \} \leq \varepsilon$$

[1] G. Calafiore and M. Campi (2004)

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A New Improved Bound[1]

- A new improved bound (based on a so-called pack-based strategy) has been recently obtained

$$N \geq \left\lceil 2/\varepsilon \log(1/2\delta) + 2n + 2n/\varepsilon \log 4 \right\rceil$$

- The main difference with the previous bound is that the factor

$$2n/\varepsilon \log(2/\varepsilon)$$

is replaced with

$$2n/\varepsilon \log 4$$

[1] T. Alamo, R. Tempo and E.F. Camacho (2007)

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RACT

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RACT

- RACT: Randomized Algorithms Control Toolbox for Matlab
- RACT has been developed at IEIIT-CNR and at the Institute for Control Sciences-RAS, based on a bilateral international project
- Members of the project

Andrey Tremba (Main Developer and Maintainer)
Giuseppe Calafiore
Fabrizio Dabbene
Elena Gryazina
Boris Polyak (Co-Principal Investigator)
Pavel Shcherbakov
Roberto Tempo (Co-Principal Investigator)

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RACT

- Main features
- Define a variety of uncertain objects: scalar, vector and matrix uncertainties, with different pdfs
- Easy and fast sampling of uncertain objects of almost any type
- Randomized algorithms for probabilistic performance verification and probabilistic worst-case performance
- Randomized algorithms for feasibility of uncertain LMIs using stochastic gradient, ellipsoid or cutting plane methods (YALMIP needed)

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Applications of Randomized Algorithms

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Application of RAs

- Randomized algorithms have been developed for various specific applications
- Control of flexible structures
- Stability and robustness of high speed networks
- Stability of quantized sampled-data systems
- Brushless DC motors
- Control design of Mini UAV

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Probabilistic Control of Mini-UAVs[1]

[1] L. Lorefice, B. Pralio and R. Tempo (2007)

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Italian National Project for Fire Prevention

- This activity is supported by the Italian Ministry for Research within the National Project

Study and development of a real-time land control and monitoring system for fire prevention

- Five research groups are involved together with a government agency for fire surveillance and patrol located in Sicily
- The aerial platform is based on the MicroHawk configuration, developed at the Aerospace Engineering Department, Politecnico di Torino, Italy

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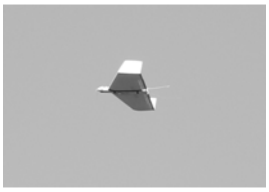
MH1000 Platform - 1

- Platform features
  - wingspan 3.28 ft (1 m)
  - total weight 3.3 lb (1.5 kg)



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MH1000 Platform - 2

- Main on-board equipment
  - various sensors and two cameras (color and infrared)
- DC motor
- Remote piloting and autonomous flight
- Flight endurance of about 40 min
- Flight envelope
  - min/max velocity: 33 ft/s (10 m/s) – 66 ft/s (17 m/s)
  - average velocity: 43 ft/s (14 m/s)

Flight Envelope (Limits)

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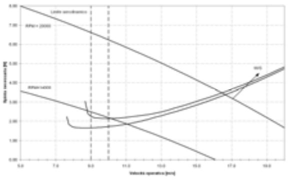
Wing loading effect → total weight	Aerodynamic constraint (red) → minimum flight speed (stall effect)
Propeller sizing effect	Propulsive constraint (blu) → maximum flight speed

velocity: 33 ft/s (10 m/s) – 66 ft/s (17 m/s)



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Basic on-board Systems

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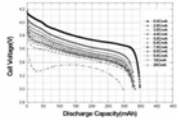
DC motor: Hacker B20-15L (4:1)
- weight: 58 g
- dimensions: Ø 20 x 40 mm
- Kv: 3700 rpm/volt

controller: Hacker Master Series 18-B-Flight
- weight: 21 g
- dimensions: 33 X 23 X 7 mm
- current drain: 18 A

battery: Kokam 2000HD (3x)
- weight: 160 g
- dimensions: 79 X 42 X 25 mm
- capacity: 2000 mAh

receiver: Schulze Alpha840W
- weight: 13.5 g
- dimensions: 52 X 21 X 13 mm
- 8 channels

servo: Graupner C1081 (2x)
- weight: 13 g
- dimensions: 23 X 9 X 21 mm
- torque: 12 Ncm



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Prototype Manufacturing - 1

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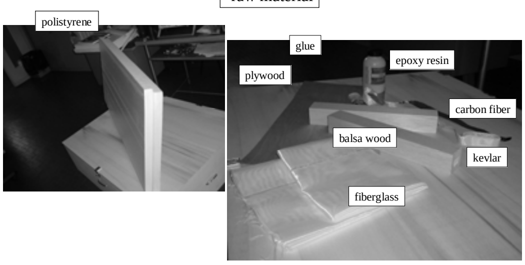
raw material



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Prototype Manufacturing - 2

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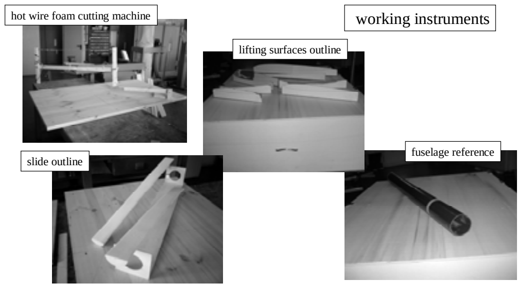
working instruments



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Prototype Manufacturing - 3

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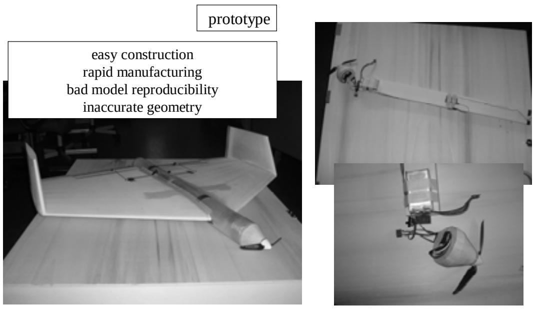
prototype



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State Space Model

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- State space formulation obtained by linearization of the full (12 coupled nonlinear ODE) model

$$\dot{x}(t) = A(\Delta)\, x(t) + B(\Delta)\, u(t)$$

$$u(t) = -\, K\, x(t)$$

where $x = [V, \alpha, q, \theta]^T$ (V flight speed, α angle of attack, q and θ pitch rate and angle), Δ uncertainty
- Consider longitudinal plane dynamics stabilization
- Control u is the symmetrical elevon deflection

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Uncertainty Description - 1

- We consider structured parameter uncertainties affecting plant and flight conditions, and aerodynamic database
- Uncertainty vector $\Delta = [\delta_1,..., \delta_{16}]$ where $\delta_i \in [\delta_i^-, \delta_i^+]$
- Key point: There is no explicit relation between state space matrices A and B and uncertainty Δ
- This is due to the fact that state space system is obtained through linearization and off-line flight simulator
- The only techniques which could be used in this case are simulation-based which lead to randomized algorithms

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Uncertainty Description - 2

- We consider random uncertainty $\Delta = [\delta_1,..., \delta_{16}]^T$
- The pdf is either uniform (for plant and flight conditions) or Gaussian (for aerodynamic database uncertainties)
- Flight conditions uncertainties need to take into account large variations on physical parameters
- Uncertainties for aerodynamic data are related to experimental measurement or round-off errors

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Plant and Flight Condition Uncertainties

parameter	pdf	$\bar{\delta}_i$	%	δ_i^-	δ_i^+	#
flight speed [ft/s]	U	42.65	± 15	36.25	49.05	1
altitude [ft]	U	164.04	± 100	0	328.08	2
mass [lb]	U	3.31	± 10	2.98	3.64	3
wingspan [ft]	U	3.28	± 5	3.12	3.44	4
mean aero chord [ft]	U	1.75	± 5	1.67	1.85	5
wing surface [ft^2]	U	5.61	± 10	5.06	6.18	6
moment of inertia [lb ft^2]	U	1.34	± 10	1.21	1.48	7

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Aerodynamic Database Uncertainties

parameter	pdf	$\bar{\delta}_i$	σ_i	#
C_X [-]	G	-0.01215	0.00040	8
C_Z [-]	G	-0.30651	0.00500	9
C_m [-]	G	-0.02401	0.00040	10
C_{Xq} [rad^{-1}]	G	-0.20435	0.00650	11
C_{Zq} [rad^{-1}]	G	-1.49462	0.05000	12
C_{mq} [rad^{-1}]	G	-0.76882	0.01000	13
C_X [rad^{-1}]	G	-0.17072	0.00540	14
C_Z [rad^{-1}]	G	-1.41136	0.02200	15
C_m [rad^{-1}]	G	-0.94853	0.01500	16

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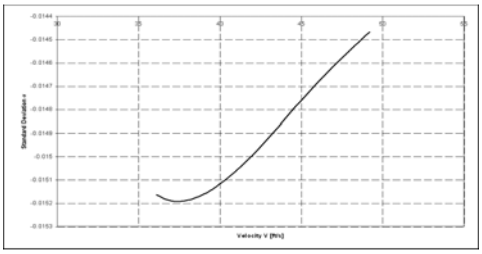
Standard Deviation and Velocity



Standard deviation is experimentally computed from the velocity

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Critical Parameters and Matrices

- We select flight speed (δ_1) and take off mass (δ_3) as critical parameters
- Flight speed is taken as critical parameter to optimize gain scheduling issues
- Take off mass is a key parameter in mission profile definition
- We define critical matrices

$$A_c^1 \quad A_c^2 \quad A_c^3 \quad A_c^4 \qquad B_c^1 \quad B_c^2 \quad B_c^3 \quad B_c^4$$

- They are constructed setting δ_1, δ_3 to the extreme values δ_1^-, δ_1^+, δ_3^-, δ_3^+ and all the remaining δ_i are equal to $\bar{\delta}_i$

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Phase 1: Random Gain Synthesis (RGS)

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- Critical parameters are flight speed and take off mass
- Specification property

  $S_1 = \{K\text{: } A_c - B_c K \textit{ satisfies the specs below}\}$

$\omega_{SP} \in [4.0,6.0]$ rad/s $\zeta_{SP} \in [0.5,0.9]$ $\omega_{PH} \in [1.0,1.5]$ rad/s
$\zeta_{PH} \in [0.1,0.3]$ $\Delta\omega_{SP} < \pm 45\%$ $\Delta\omega_{PH} < \pm 20\%$

where ω and ζ are undamped natural frequency and damping ratio of the characteristic modes; $_{SP}$ and $_{PH}$ denote short period and phugoid mode

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Specs in the Complex Plane

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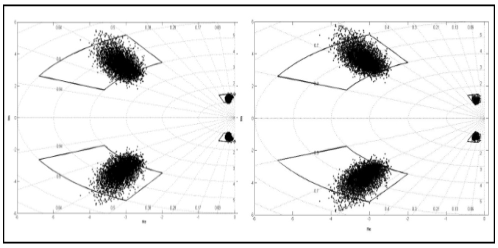

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Volume of the Good Set

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- Define a bounding set B of gains K

$$B = \{K\text{: } k_i \in [k_i^-, k_i^+],\ i = 1,\ldots,4\}$$

- Define the volume of the *good set*

$$\mathrm{Vol}_{good} = \int_A \mathrm{d}\,K$$

  where $A = \{K \in B \cap S_1\}$

- Vol_B is simply the volume of the hyperrectangle B

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Randomized Algorithm 1 (RGS)

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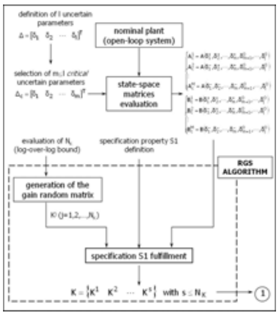
- Uniform pdf for controller gains K in given intervals
- Accuracy and confidence $\varepsilon = 4 \cdot 10^{-5}$ and $\delta = 3 \cdot 10^{-4}$
- Number of random samples is computed with “Log-over-Log” Bound obtaining $N = 200{,}000$
- We obtained 5 gains K^i satisfying specification property S_1



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Randomized Algorithm 1 (RGS)

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Given ε, $\delta \in (0,1)$, RGS *returns the set of gains* $\{K^1,\ldots,K^s\}$ *satisfying* S_1

1. Compute N using the Log-over-log Bound;
2. For fixed j=1,2,…,N, generate uniformly the gain random matrix $K^j \in B$;
3. Set C=0;
4. For fixed i=1,2,3,4, compute the closed-loop matrix
   $A_{cl}{}^i(K^j) = A_c{}^i - B_c{}^i K^j$;
   - if $K^j \in S_1$, set $C = C$+1;
   - otherwise, set $C = C$;
5. End;
6. If C = 4, return the gain K^j;
7. Set $j = j$+1 and return to Step 2;
8. End

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Random Gain Set

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gain set	K_V	K_α	K_q	K_θ
K^1	0.00044023	0.09465000	0.01577400	-0.00473510
K^2	0.00021450	0.09581200	0.01555500	-0.00323510
K^3	0.00054999	0.09430800	0.01548200	-0.00486340
K^4	0.00010855	0.09183200	0.01530000	-0.00404380
K^5	0.00039238	0.09482700	0.01609300	-0.00417340

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Phase 2: Random Stability Robustness Analysis (RSRA)

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- Take $K_{rand} = K^i$ obtained in Phase 1
- Randomize Δ according to the given pdf and take N random samples Δ^i
- Specification property

$$S_2 = \{\Delta: A(\Delta) - B(\Delta)\ \ K_{rand} \textit{ satisfies the specs of } S_1\}$$

- Computation of the empirical probability of stability $\hat{p}_N$

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Empirical Probability

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- Consider fixed gain K_{rand}
- Define the probability

$$p_{true} = \int_C p(\Delta)\, d\,\Delta$$

where $C = \{\Delta \in B \cap S_2\}$ and $p(\Delta)$ is the given pdf
- Then, we introduce a "success" indicator function

$$I(\Delta^j) = 1 \text{ if } \Delta^j \in S_2$$

or $I(\Delta^j) = 0$ otherwise
- The empirical probability for S_2 is given by

$$\hat{p}_N = N_{good}/N$$

where N_{good} is equal to the number of successes

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Randomized Algorithm 2 (RSRA)

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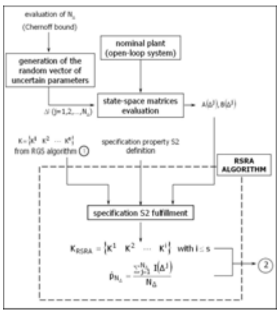
- Take K_{rand} from Phase 1
- Accuracy and confidence
  $\varepsilon = \delta = 0.0145$
- Number of random samples is computed with Chernoff Bound obtaining $N = 5{,}000$
- Empirical probability is defined using an indicator function



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Randomized Algorithm 2 (RSRA)

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Given $\varepsilon, \delta \in (0,1)$, RSRA *returns the empirical probability* $\hat{p}_N$ *that* S_2 *is satisfied for a gain* K_{rand} *provided by Algorithm 1*

1. Compute N using the Chernoff Bound;
2. Generate N random vectors $\Delta^j \in B$ according to the given pdf;
3. For fixed $j=1,2,\ldots,N$, compute the closed-loop matrix
   $$A_{cl}(\Delta^j) = A(\Delta^j) - B(\Delta^j)K_{rand};$$
   - if $A_{cl}(\Delta^j) \in S_2$, set $I(\Delta^j) = 1$;
   - otherwise, set $I(\Delta^j) = 0$;
4. End;
5. Return the empirical probability $\hat{p}_N$

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Empirical Probability of Stability for Phase 2

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gain set	empirical probability
K^1	88.56%
K^2	90.60%
K^3	89.31%
K^4	93.86%
K^5	85.14%

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Probability Degradation Function

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- Flight condition uncertainties are multiplied by the *amplification factor* $\rho > 0$ keeping the nominal value constant

$$\delta_i \in \rho\,[\delta_i^-, \delta_i^+] \quad \text{for } i = 1, 2, \ldots, 7$$

- No uncertainty affects the aerodynamic database, i.e.

$$\delta_i = \bar{\delta_i} \quad \text{for } i = 8, 9, \ldots, 16$$

- For fixed $\rho \in [0,1.5]$ we compute the empirical probability for different gain sets K^i
- The plot empirical probability vs ρ is the probability degradation function

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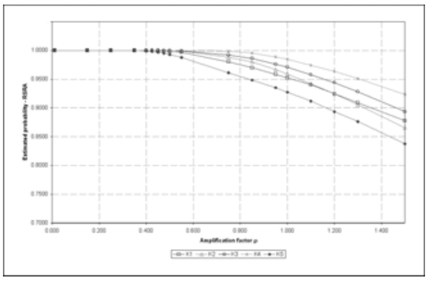
Probability Degradation Function for Phase 2



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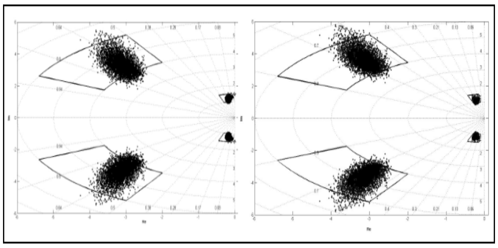
Root Locus Plot for Phase 2



Root locus for K^2 (left) and K^4 (right)

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Phase 3: Random Performance Robustness Analysis (RPRA)

- This phase is similar to Phase 2, but military specs are considered (bandwidth criterion)
- Specification property

  $S_3 = \{\Delta: A(\Delta) - B(\Delta)\ \ K_{rand}$ *satisfies the specs below*$\}$

  $\omega_{BW} \in [2.5,5.0]$ rad/s $\qquad \tau_P \in [0.0,0.5]$ s

  where ω_{BW} and τ_P are bandwidth and phase delay of the frequency response
- Computation of the empirical probability that S_3 is satisfied

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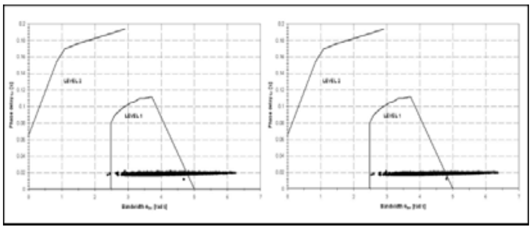
Bandwidth Criterion



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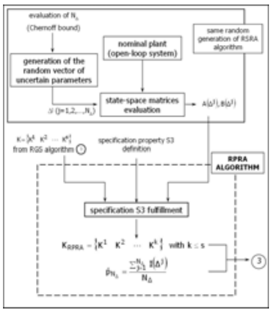
Randomized Algorithm 3 (RPRA)

- Take K_{rand} from Phase 1
- Numer of random samples is computed with the Chernoff Bound obtaining N =5,000
- Empirical probability is defined using an indicator function



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Randomized Algorithm 3 (RPRA)

Given N and $A_{cl}(\Delta^j)$, j=1,2,…,N, *provided by Algorithm 2,* RPRA *returns the empirical probability* $\hat{p}_N$ *that* S_3 *is satisfied for a gain* K_{rand} *provided by Algorithm 1*

1. For fixed j=1,2,…,N
   - if $A_{cl}(\Delta^j) \in S_3$, set $I(\Delta^j) = 1$;
   - otherwise, set $I(\Delta^j) = 0$;
2. End;
3. Return the empirical probability $\hat{p}_N$

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Empirical Probability of Performance for Phase 3

gain set	empirical probability
K^1	93.58%
K^2	95.16%
K^3	90.80%
K^4	84.78%
K^5	96.06%

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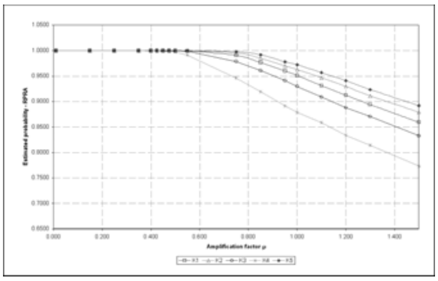
Probability Degradation Function for Phase 3



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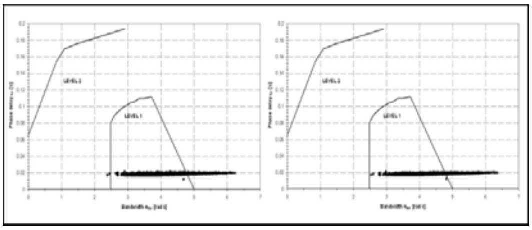
Bandwidth Criterion for Phase 3



Bandwidth criterion for K^1 (left) and K^3 (right)

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Gain Selection

- Multi-objective criterion as a compromise between different specifications

Finally we selected gain K^1 as the best compromise between all the specs and criteria!

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Conclusions: Flight Tests in Sicily - 1

- Evaluation of the payload carrying capabilities and autonomous flight performance
- Mission test involving altitude, velocity and heading changing was performed in Sicily
- Checking effectiveness of the control laws for longitudinal and lateral-directional dynamics
- Flight control design based on RAs for stabilization and guidance

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Conclusions: Flight Tests in Sicily - 2

- Satisfactory response of MH1000
- Possible improvements by iterative design procedure
- Stability of the platform is crucial for the video quality and in the effectiveness of the surveillance and monitoring tasks

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Color Camera: Right Turn

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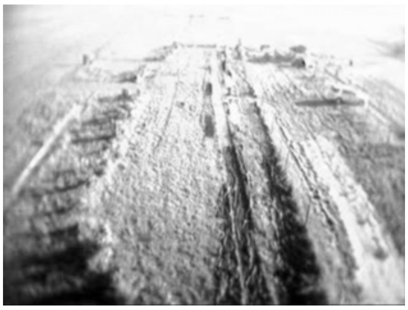
Color Camera: Landing Phase



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Infrared Camera - 1

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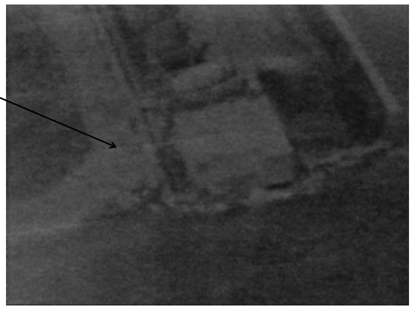
Infrared Camera - 1

road



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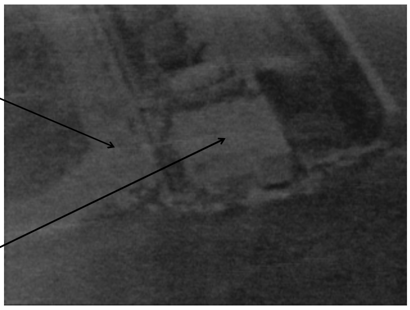
Infrared Camera - 1

road



shed

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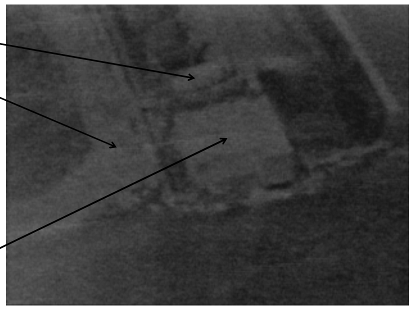
Infrared Camera - 1

car

road



shed

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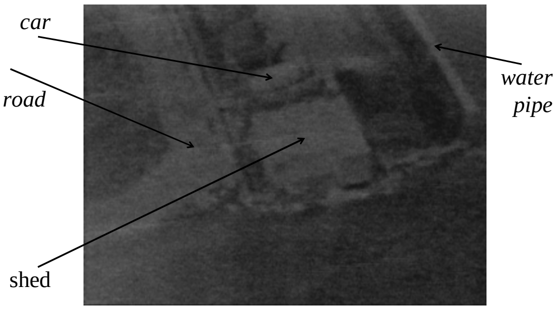
Infrared Camera - 1



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Infrared Camera - 2

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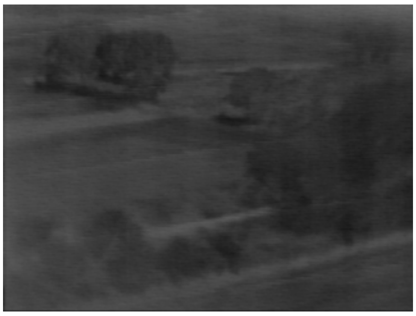
Infrared Camera - 3



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Acknowledgment

- Thanks to Andrea Sanna and his students for the flight test video and the computer graphics animation

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Conclusions

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PAC Algorithms

- Randomized algorithms are Probably Approximately Correct (PAC)
- We give up a guaranteed deterministic solution
- This implies accepting a “small” risk of giving a wrong solution
- The risk can be made arbitrarily small (but not zero) taking suitable values of so-called confidence and accuracy

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PAC Algorithms

- Two open problems

- Optimization with sequential methods
- Derive “reasonable” bounds for the statistical learning theory approach

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